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PROCEEDINGS

OF THE

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PROCEEDINGS

OF THE

EDINBURGH MATHEMATICAL SOCIETY.

THIRTEENTH SESSION, 1894–95.

First Meeting, November 9th, 1894.

Prof. KNOTT, D.Sc., F.R.S.E., President, in the Chair.

For this Session the following Office-bearers were elected :—

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Étude sur le Triangle et sur certains points de Géométrographie.

PAR M. ÉMILE LEMOINE.

Cette note que j'ai l'honneur de presenter à la Société Mathématique d'Edinburgh, par l'entremise aimable de M J. S Mackay, contient, ou des résultats que je crois nouveaux, ou des *développements* sur des sujets que j'ai déjà souvent abordés dans la Géométrie et qui concernent : *la transformation continue* dans le triangle et dans le tétraèdre, *les formules entre les éléments du triangle*, et la *Géométrographie*. Pour abréger, je passerai rapidement sur les points que j'ai déjà développés ailleurs, me contentant de renvoyer, si l'on désire plus d'explications, aux mémoires où la chose a été faite.

L'idée de la *transformation continue* n'est autre qu'une explicitation, pour le triangle et pour le tétraèdre, du principe de continuité de Carnot ; elle permet de transformer les théorèmes, les formules, les équations qui déterminent les éléments de ces figures ou expriment leurs propriétés, de façon à ce que, sous le nouveau vêtement qu'elle leur fait prendre, on obtient des théorèmes, des formules, des équations nouvelles, mais qui ne sont, au fond, que des formes d'une même vérité.

Notations pour le triangle ABC et ses éléments.

Nous nommons a, b, c ; A, B, C ; $p, p-a, p-b, p-c$; S ; R ;

[redacted]

Quantités.	Transformées en A.	Transformées en B.	Transformées en C.
a, b, c	$a, -b, -c$	$-a, b, -c$	$-a, -b, c$
A, B, C, ω	$-A, \pi-B, \pi-C, -\omega$	$\pi-A, -B, \pi-C, -\omega$	$\pi-A, \pi-B, -C, -\omega$
$p, (p-a), (p-b), (p-c)$	$-(p-a), -p, (p-c), (p-b)$	$-(p-b), (p-c), -p, (p-a)$	$-(p-c), (p-b), (p-c), -p$
S, R	$-S, -R$	$-S, -R$	$-S, -R$
r, r_a, r_b, r_c	$r_a, r, -r_c, -r_b$	$r_b, -r_c, r, -r_a$	$r_c, -r_b, -r_a, r$
$\delta, \delta_a, \delta_b, \delta_c$	$-\delta_a, -\delta, -\delta_c, -\delta_b$	$-\delta_b, -\delta_c, -\delta, -\delta_a$	$-\delta_c, -\delta_b, -\delta_a, -\delta$
h_a, h_b, h_c	$-h_a, h_b, h_c$	$h_a, -h_b, h_c$	$h_a, h_b, -h_c$
l_a, l_b, l_c	$-l_a, -l'_b, -l'_c$	$-l'_a, -l_b, -l'_c$	$-l'_a, -l'_b, -l_c$
l'_a, l'_b, l'_c	$l'_a, -l_b, -l_c$	$-l_a, l'_b, -l_c$	$-l_a, -l_b, l'_c$
x, y, z	$-x, y, z$	$x, -y, z$	$x, y, -z$
α, β, γ	α, β, γ	α, β, γ	α, β, γ
X, Y	$X, -Y$	$-X, Y$	$-X, -Y$

Si, dans une formule ou dans une équation se rapportant au triangle ABC et contenant les éléments marqués dans la première colonne de ce tableau indiquée par le mot : quantité, je remplace ces éléments par ceux qui leur correspondent dans une des trois colonnes suivantes, j'aurai opéré respectivement la *transformation continue* en A, en B, ou en C.

Dans une *transformation continue* :

I. La droite de l'infini se transforme en elle-même ; donc, deux droites parallèles restent parallèles dans la figure transformée.

II. Les points circulaires de l'infini se transforment l'un dans l'autre.

III. Le degré et la classe des courbes se conservent dans les courbes transformées.

IV. L'homographie, l'homologie, et l'orthologie ainsi que l'involution, se conservent.

V. Deux droites ou deux courbes qui sont orthogonales restent orthogonales.

VI. Si les longueurs de deux droites sont dans un rapport numérique indépendant des éléments du triangle, ce rapport se [redacted]

b. A chacune des trois transformations correspond un résultat différent.

Exemples : Au point qui a pour coordonnées

$$\frac{p-a}{a},\ \frac{p-b}{b},\ \frac{p-c}{c}\ \text{(point de Nagel)}$$

correspondent respectivement les trois points :

$$-\frac{p}{a},\ \frac{p-c}{b},\ \frac{p-b}{c};\quad \frac{p-c}{a},\ -\frac{p}{b},\ \frac{p-a}{c};$$

$\frac{p-b}{a},\ \frac{p-a}{b},\ -\frac{p}{b}$, avec des propriétés analogues à celles du point de Nagel.

L'axe anti-orthique (droite qui passe par les pieds des trois bissectrices extérieures et a pour équation $x+y+z=0$) devient par *transformation continue* en A, la droite qui passe par les pieds des bissectrices intérieures partant de B et de C et par le pied de la bissectrice extérieure partant de A ; son équation est $-x+y+z=0$. On appelle souvent cette droite l'interbissectrice relative à A. Résultats analogues par transformation en B et en C.

c. Une des transformations conserve le résultat primitif, les deux autres le changent en un autre résultat, mais unique pour ces deux transformations.

Exemple : La formule $S=\frac{ar_b r_c}{\delta - r_a}$ se reproduit par transformation en A, mais transformée soit en B, soit en C, elle donne

$$S=\frac{arr_a}{4R-r_b-r_c}=\frac{arr_a}{r_a-r}$$

d. La transformation continue faite soit en A, *soit en* B, *soit en* C *donne un même résultat différent de celui qu'on transforme.*

Exemple : La conique inscrite qui a pour un de ses foyers le point d'où l'on voit les trois côtés sous le même angle (Conique de Simmons : voir J. J. Milne, *Companion to the Weekly Problem Papers*, p. 165), conique qui a pour équation

$$\sqrt{x\sin(A+60^\circ)} + \sqrt{y\sin(B+60^\circ)} + \sqrt{z\sin(C+60^\circ)} = 0,$$

se transforme soit en A, soit en B, soit en C, en

$$\sqrt{x\sin(A-60^\circ)} + \sqrt{y\sin(B-60^\circ)} + \sqrt{z\sin(C-60^\circ)} = 0.$$

Je n'ai pas trouvé de cas où l'une des transformations reproduise la formule et où les deux autres la modifient mais chacune d'une façon différente.

La transformation continue s'applique au tétraèdre. Nous allons donner seulement le tableau que permet la transformation des éléments de la figure.

NOTATIONS.

1°. *Je désigne par* A, B, C, D *les sommets du tétraèdre.*

2°. *Les faces* ABC, BCD, CDA, DAB *seront* F_d, F_a, F_b, F_c, *nous poserons* $F_a + F_b + F_c + F_d = S$.

3°. *Les angles plans des faces*

[redacted]

Il y aura donc les douze angles

$$\widehat{aF_c},\quad \widehat{aF_b},\quad \widehat{a'F_d},\quad \widehat{a'F_a}$$
$$\widehat{bF_c},\quad \widehat{bF_a},\quad \widehat{b'F_d},\quad \widehat{b'F_b}$$
$$\widehat{cF_b},\quad \widehat{cF_a},\quad \widehat{c'F_d},\quad \widehat{c'F_c}$$

6°. *Les hauteurs seront* : h_a, h_b, h_c, h_d.

7°. *Les rayons et les centres de la sphère inscrite et des sphères ex-inscrites de* PREMIÈRE ESPÈCE *seront* :

$$r, r_a, r_b, r_c, r_d\ ;\quad o, o_a, o_b, o_c, o_d.$$

Les rayons et les centres des sphères ex-inscrites de SECONDE ESPÈCE, *ou sphères inscrites dans les combles du tétraèdre seront*

$$r_a', r_b', r_c'\ ;\quad o_a', o_b', o_c',$$

r_a' et o_a' ; r_b' et o_b' ; r_c' et o_c' appartenant *respectivement* à la sphère inscrite dans l'un des combles qui ont pour arêtes DA ou BC ; DB ou CA ; DC ou AB.

(On sait qu'il n'y a qu'une seule sphère pour deux combles opposés.)

Le tétraèdre *général* possède toujours ces huit sphères tangentes aux quatre faces ; quand une sphère ou deux sphères, ou les trois sphères inscrites dans les combles manquent, ce qui peut arriver, car lorsque la somme de deux faces qui ont même arête est égale à la somme des deux autres, la sphère des combles correspondant à ces arêtes a un rayon infini ; ce ne sont plus alors des tétraèdres *généraux* puisqu' il y a une ou plusieurs relations entre les faces, et la *transformation continue* n'est plus applicable, au moins sans discussion préalable.

8°. *Le volume du tétraèdre et le rayon de la sphère circonscrite seront* : V et R.

O sera le centre de cette sphère.

9°. *Les angles de* DA *avec* BC ; *de* DB *et de* AC ; *de* DC *et de* BA *seront* : α, β, γ.

10°. *Les longueurs des droites qui joignent les milieux de* DA *et de* BC ; *de* DB *et de* AC *de* DC *et de* BA *seront* : l, m, n.

De même que la *transformation continue* en A, dans le triangle, revient à changer a, b, c en $a, -b, -c$, la *transformation continue* en D, dans le tétraèdre, revient à changer a, b, c, a', b', c' en $a, b, c, -a', -b', -c'$.

[redacted]

[redacted] Transformées en D	Transformées en A.
[redacted] $, -a', -b', -c'$	a, b, c, a', b', c'
[redacted] $F_b, -F_c, F_d$	$F_a, -F_b, -F_c, -F_d$
[redacted] $, -D_b, -D_c$	$D_a, \pi - D_b, \pi - D_c$
[redacted] $\pi - A_c, \pi - A_b$	$-A_d, -A_c, -A_b$
[redacted] $-B_c, \pi - B_a$	$\pi - B_d, \pi - B_c, B_a$
[redacted] $\pi - C_a, \pi - C_b$	$\pi - C_d, C_a, \pi\ C_b$
[redacted] $b, \pi\ c, a', b', c'$	$\pi - a, b, c, a', \pi - b', \pi - \widehat{c'}$
[redacted] $h_b, h_c, -h_d$	$-h_a, h_b, h_c, h_d$
[redacted] $\widehat{a'F_a}, \widehat{aF_b}, \widehat{aF_c}$	$-a'F_d, \pi - a'F_a, aF_b, \widehat{aF_c}$
[redacted] $\widehat{bF_a}, \widehat{b'F_b}, \widehat{bF_c}$	$b'F_d, \pi - bF_a, b'F_b, -\widehat{bF_c}$
[redacted] $\widehat{cF_a}, \widehat{cF_b}, \widehat{c'F_c}$	$c'F_d, \pi - cF_a, cF_b, \widehat{c'F_c}$
[redacted] r'_a, r'_b, r'_c, r	$r_c, r, -r'_c, -r'_b, r'_a$
[redacted] r_a, r_b, r_c	$r_d, -r, -r_b$

Quantités.	Transformées en B.	Transformées en C.
a, b, c, a', b', c'	$-a, b, -c, a', -b', c'$	$-a, -b, c, a', b', -c'$
F_a, F_b, F_c, F_d	$-F_a, F_b, -F_c, -F_d$	$-F_a, -F_b, F_c, -F_d$
D_a, D_b, D_c	$\pi - D_a, D_b, \pi - D_c$	$\pi - D_a, \pi - D_b, D_c$
A_d, A_c, A_b	$\pi - A_d, \pi - A_c, A_b$	$\pi - A_d, A_c, \pi - A_b$
B_d, B_c, B_a	$-B_d, -B_c, -B_a$	$\pi - B_d, B_c, \pi - B_a$
C_d, C_a, C_b	$\pi - C_d, \pi - C_a, C_b$	$-C_d, -C_a, -C_b$
$\hat{a}, \hat{b}, \hat{c}, \hat{a'}, \hat{b'}, \hat{c'}$	$\hat{a}, \pi - \hat{b}, \hat{c}, \pi - \hat{a'}, \hat{b'}, \pi - \hat{c'}$	$\hat{a}, \hat{b}, \pi - \hat{c}, \pi - \hat{a'}, \pi - \hat{b'}, \hat{c'}$
h_a, h_b, h_c, h_d	$h_a, -h_b, h_c, h_d$	$h_a, h_b, h_c, -h_d$
$\widehat{a'F}_d, \widehat{a'F}_a, \widehat{aF}_b, \widehat{aF}_c$	$\widehat{a'F}_d, \widehat{a'F}_a, \pi - \widehat{aF}_b, -\widehat{aF}_c$	$\widehat{a'F}_d, \widehat{a'F}_a, -\widehat{aF}_b, \pi - \widehat{aF}_c$
$\widehat{b'F}_d, \widehat{bF}_a, \widehat{b'F}_b, \widehat{bF}_c$	$-\widehat{b'F}_d, \widehat{bF}_a, \pi - \widehat{b'F}_b, \widehat{bF}_c$	$\widehat{b'F}_d, -\widehat{bF}_a, \widehat{b'F}_b, \pi - \widehat{bF}_c$
$\widehat{c'F}_d, \widehat{cF}_a, \widehat{cF}_b, \widehat{c'F}_c$	$\widehat{c'F}_d, -\widehat{cF}_a, \pi - \widehat{cF}_b, \widehat{c'F}_c$	$-\widehat{c'F}_d, \widehat{cF}_a, \widehat{cF}_b, \pi - \widehat{c'F}_c$
r, r_a, r_b, r_c, r_d	r_b, r'_c, r, r'_a, r'_b	$r_c, -r'_b, -r'_a, r, r'_c$
r'_a, r'_b, r'_c	$-r_c, r_d, -r_a$	$-r_b, -r_a, r_d$

V et R se changent en : $-$V et $-$R dans les quatre transformations.

α, β, γ deviennent : $\pi - \alpha, \pi - \beta, \pi - \gamma$ dans les quatre transformations.

l, m, n ne changent pas.

Nous avons vu par le tableau relatif au triangle qu'à un point donné $M(x, y, z)$ dans le plan du triangle peuvent correspondre trois *transformés continus* M_a, M_b, M_c dont les coordonnées sont

$$-x_a,\ y_a,\ z_a\ ;\quad x_b,\ -y_b,\ z_b\ ;\quad x_c,\ y_c,\ -z_c$$

en affectant des indices a, b, c les quantités qui représentent ce que deviennent les coordonnées quand on les transforme en A, en B, et en C. Par rapport au tétraèdre un point $M(x, y, z, t)$ peut avoir sept transformés continus dont les coordonnées sont

1°. Quatre transformés de première espèce

$$x_d,\ y_d,\ z_d,\ -t_d\ ;\quad -x_a,\ y_a,\ z_a,\ t_a\ ;\quad x_b,\ -y_b,\ z_b,\ t_b\ ;\quad x_c,\ y_c,\ -z_c,\ t_c$$

2°. Trois transformés de seconde espèce

$$-x_{da},\ y_{da},\ z_{da},\ -t_{da}\ ;\quad x_{db},\ -y_{db},\ z_{db},\ -t_{db}\ ;\quad x_{dc},\ y_{dc},\ -z_{dc},\ -t_{dc}$$

x_d désignant ce que devient x par *transformation continue* en D, x_{da} désignant ce que devient x si l'on fait d'abord la *transformation continue* en D sur lui, ce qui donne x_d ; puis la *transformation continue* en A sur x_d, ce qui donne x_{da}, etc.

Pour les démonstrations et l'exposition de la *transformation continue* nous renvoyons aux mémoires suivants : *Association Française pour l'avancement des Sciences, Congrès de Marseille* 1891 ; [redacted]

une manière de traiter le célèbre problème d'Apollonius, manière qui, je crois, n'a pas encore été considérée. Pour fixer les idées nous supposons, sur la figure (Fig. 1) qu'il s'agit de chercher le rayon ρ du cercle de centre o qui a les trois cercles donnés à l'extérieur.

Joignons oA, oB, oC que nous appelons X, Y, Z ; pour tout point du plan on a la relation

$$\Sigma a^2X^4 \quad \Sigma(b^2+c^2-a^2)(a^2X^2+Y^2Z^2)+a^2b^2c^2=0$$

entre les distances X, Y, Z d'un point quelconque aux trois sommets.

D'ailleurs, comme on a :

$$X=R_a+\rho,\ Y=R_b+\rho,\ Z=R_c+\rho$$

on peut écrire

$$\Sigma a^2(\rho+R_a)^4-\Sigma(b^2+c^2+c^2)\{a^2(\rho+R_a)^2+(\rho+R_b)^2(\rho+R_c)^2\}+a^2b^2c^2=0$$

Si l'on développe cette équation en l'ordonnant par rapport à ρ, on voit très facilement que les coëfficients des termes en ρ^4 et en ρ^3 sont nuls identiquement et il vient

$$\begin{aligned}(1)\qquad &\rho^2\Sigma[6a^2R_a^2-(b^2+c^2-a^2)\{a^2+(R_b+R_c)^2+2R_bR_c\}]\\ &+2\rho\,\Sigma[2a^2R_a^3-(b^2+c^2-a^2)\{a^2R_a+R_b\,R_c\,(R_b+R_c)\}]\\ &+a^2b^2c^2+\Sigma\,\{a^2R_a^4-(b^3+c^2-a^4)\,(a^2R_a^2+R_b^2R_c^2)\}=0\end{aligned}$$

Cette équation donnera deux valeurs ρ' et ρ'' correspondant aux cercles qui touchent les trois cercles donnés en les ayant tous les trois à l'extérieur, ou tous les trois à l'intérieur.

Pour revenir au problème que je me proposais de résoudre, il faut faire : $R_a=a,\ R_b=b,\ R_c=c$

et calculer le coëfficient de ρ^2, celui de ρ, et le terme indépendant de ρ. Appelons L, M, N ces coëfficients, il faut évaluer maintenant

$$\begin{aligned}L&=\Sigma[6a^4-(b^2+c^2-a^2)\{a^2+(b+c)^2+2bc\}]\\ M&=\Sigma[2a^5-(b^2+c^2-a^2)\{a^3+bc(b+c)\}]\\ N&=a^2b^2c^2+\Sigma\{a^6-(b^2+c^2-a^2)(a^4+b^2c^2)\}.\end{aligned}$$

Calcul de L. On peut écrire

$$L = 6\Sigma a^4 - \Sigma(a^2+b^2+c^2-2a^2)(a^2+b^2+c^2+4bc), \text{ ou}$$

$$L = 6\Sigma a^4 - \Sigma(a^2+b^2+c^2)^2 + 2\Sigma a^2(a^2+b^2+c^2) - 4\Sigma bc(a^2+b^2+c^2) + 8\Sigma a^2bc, \text{ ou}$$

$$L = 6\Sigma a^4 - 3(a^2+b^2+c^2)^2 + 2(a^2+b^2+c^2)\Sigma a^2 - 4(a^2+b^2+c^2)\Sigma bc + 8abc\Sigma a, \text{ ou}$$

$$L = 6\Sigma a^4 - 2(a^2+b^2+c^2)^2 - 4(a^2+b^2+c^2)\Sigma bc + 16pabc.$$

Mais les formules dont je parlais tout à l'heure (voir *Mathesis* 1892, loco citato) donnent

$$\Sigma a^4 = 2\{(p^2-r\delta)^2 - 4S^2\}, \quad a^2+b^2+c^2 = 2(p^2-r\delta)$$

$$\Sigma bc = p^2 + r\delta\,;$$

on a d'ailleurs, $abc = 4RS$ et $S = pr$.

Substituant et effectuant les calculs, on trouve très facilement

$$L = 16r^2(\delta^2 - 4p^2)$$

Calcul de M. On a :

$$M = 2\Sigma a^4 - \Sigma(b^2+c^2-a^2)a^2 - \Sigma bc(b+c)(b^2+c^2-a^2), \text{ ou}$$

[redacted]

Calcul de N. On a

$$N = a^2b^2c^2 + \Sigma\{a^6 - (b^2 + c^2 - a^2)(a^4 + b^2c^2)\} \quad \text{ou}$$

$$N = a^2b^2c^2 + \Sigma a^6 - \Sigma(a^2 + b^2 + c^2 - 2a^2)(a^4 + b^2c^2) \quad \text{ou}$$

$$N = a^2b^2c^2 + \Sigma a^6 - (a^2 + b^2 + c^2)\Sigma a^4 + 2\Sigma a^6 - (a^2 + b^2 + c^2)\Sigma b^2c^2 + 2\Sigma a^2b^2c^2 \quad \text{ou}$$

$$N = 7a^2b^2c^2 + 3\Sigma a^6 - (a^2 + b^2 + c^2)\Sigma a^4 - (a^2 + b^2 + c^2)\Sigma b^2c^2$$

On trouve dans nos formules

$$\Sigma b^2c^2 = (p^2 - r\delta)^2 + 4S^2$$

On n'y trouve pas Σa^6 mais il peut se calculer aisément en partant de Σa^4 et de Σa^2.

On a en effet

$$\begin{aligned}(a^4 + b^4 + c^4)(a^2 + b^2 + c^2) &= \Sigma a^6 + \Sigma a^2(b^4 + c^4)\\ &= \Sigma a^6 + \Sigma b^2c^2(b^2 + c^2) \quad \text{ou}\end{aligned}$$

$$\begin{aligned}4\{(p^2 - r\delta)^2 - 4S^2\}(p^2 - r\delta) &= \Sigma a^6 + \Sigma b^2c^2(a^2 + b^2 + c^2 - a^2)\\ &= \Sigma a^6 + 2(p^2 - r\delta)\Sigma b^2c^2 - 3a^2b^2c^2 \quad \text{ou}\end{aligned}$$

$$\begin{aligned}4(p^2 - r\delta)^3 - 16p^2r^2(p^2 - r\delta) &= \Sigma a^6 + 2(p^2 - r\delta)\{(p^2 - r\delta)^2 + 4S^2\}\\ &\quad - 48p^2R^2r^2\end{aligned}$$

d'où $$\Sigma a^6 = 2(p^2 - r\delta)^3 - 24p^2r^2(p^2 - r\delta) + 48p^2R^2r^2$$

Substituant on trouve

$$N = 64p^2r^2\{(2R + r)^2 - p^2\}$$

Remarquons, en passant, que l'on a

$$p^2 - (2R + r)^2 = 4R^2\cos A\cos B\cos C.$$

L'équation qui détermine ρ devient alors

$${}^2(\delta^2 - 4p^2) + 4p\rho\{\delta(2R + r) - 2p^2\} - 4p^2\{p^2 - (2R + r)^2\} = 0$$

On en tire, toutes réductions faites :

$$\rho' = \frac{2p(2R + r - p)}{2p - \delta} \qquad \rho'' = -\frac{2p(2R + r + p)}{2p + \delta} \tag{2}$$

La formule (1) lorsque les racines sont réelles donne toujours les rayons avec un signe, naturellement ; mais il faut interpréter géométriquement ce signe. Ainsi, avec les deux formules précédentes, si le triangle est équilatéral, on trouve

$$\rho' = -\frac{\sqrt{3} - 1}{\sqrt{3}}, \quad \rho'' = -\frac{\sqrt{3} + 1}{\sqrt{3}}$$

et ce sont bien, mais en *valeur absolue* seulement, les valeurs des rayons qui conviennent pour ce cas, comme la géométrie le montre immédiatement. (Voir la note additionnelle à la fin du mémoire.)

Si l'on applique la *Transformation continue* en A aux formules (2), il vient

$$\rho_a' = \frac{2(p - a)\{-2R + r_a + (p - a)\}}{2(p - a) - \delta_a},$$

$$\rho_a'' = -\frac{2(p - a)\{-2R + r_a + (p - a)\}}{2(p - a) + \delta_a}$$

[redacted]

Remarques.

L'axe de similitude externe des trois circonférences

$$A(a),\ B(b),\ C(c) \text{ est la droite } a^2x + b^2y + c^2z = 0.$$

On en conclut, par *transformation continue* en A, que la droite (coordonnées normales) $-a^2x + b^2y + c^2z = 0$ est l'axe de similitude qui passe par les centres de similitude interne de $A(a)$ et $B(b)$, et de $A(a)$ et $C(c)$. Nous désignons par $M(R)$ un cercle de rayon R et de centre M.

Le centre radical de ces trois circonférences est le point dont les coordonnées normales sont :

$$\cos A - \cos B \cos C,\quad \cos B - \cos C \cos A,\quad \cos C - \cos A \cos B$$

c'est un point que l'on rencontre assez souvent dans la Géométrie du triangle et qui est le symétrique de l'orthocentre par rapport au centre du cercle circonscrit.

On trouve très simplement les coordonnées de ce centre radical ; en effet (Association française, 1888, Congrès d'Oran, p. 170 vi.) il a pour coordonnées normales

$$abc\cos A - a^3 + b^3\cos C + c^3\cos B, \text{ etc.}$$

Mais $$b^3\cos C + c^3\cos B - a^3 = 4RS(\cos A - 2\cos B\cos C)\,;$$

elles deviennent donc

$$4RS\cos A + 4RS(\cos A - 2\cos B\cos C), \text{ etc.}$$

ou $\cos A - \cos B\cos C$, comme nous l'avons dit.

La remarque permet de placer très simplement ce point dans le triangle ABC.

Pour cela je trace les trois cercles $A(a)$, $B(b)$, $C(c)$ op : $(9C_1 + 3C_3)$

et comme ces cercles se rencontrent deux à deux, il suffit de tracer deux de leurs intersections op : $(4R_1 + 2R_2)$

qui se coupent au point cherché.

En tout op : $(4R_1 + 2R_2 + 9C_1 + 3C_3)$

Simplicité 18 ; exactitude 13 ; 2 droites, 3 cercles.

Sans entrer dans d'autres détails j'énoncerai encore les applications suivantes de la transformation continue et de nos formules.

Les points ω, ω' qui ont pour coordonnées normales respectivement

$$\frac{a+r_a}{a},\ \frac{b+r_b}{b},\ \frac{c+r_c}{c};\qquad \frac{a-r_a}{a},\ \frac{b-r_b}{b},\ \frac{c-r_c}{c}$$

sont des points qui jouissent de propriétés remarquables et que j'ai souvent rencontrés, ainsi que leurs transformés continus en A : ω_a, ω_a'

$$\frac{a+r}{a},\ \frac{b+r_c}{b},\ \frac{c+r_b}{c};\qquad \frac{a-r}{a},\ \frac{b-r_c}{b},\ \frac{c-r_b}{c};$$

en B : ω_b, ω_b', etc.

Ces huit points $\omega, \omega', \omega_a, \omega_a'$, etc., sont les centres des quatre couples de cercles tangents savoir :

ω, ω' aux cercles $A(p-a)$, $B(p-b)$, $C(p-c)$ tangents deux à deux ;

ω_a, ω_a' aux cercles $A(p)$, $B(p-c)$, $C(p-a)$;

ω_b, ω_b' etc.

[redacted]

On a : $$\overline{\omega\,\omega'}^2 = 16S^2\frac{\delta^2 - 3p^2}{(4p^2 - \delta^2)^2}\ ;$$

et par transformation continue en A

$$\overline{\omega_a\omega_a'}^2 = 16S^2\frac{\delta_a^2 - 3(p-a)^2}{\{4(p-a)^2 - \delta_a^2\}^2}$$

Les coordonnées cartésiennes du centre O du cercle inscrit par rapport à HB pris pour axe des x et à HA pour axe des y sont, comme il est facile de le voir, (H étant l'orthocentre)

$$x = \frac{c\cos B - (p-b)}{\sin C} \qquad y = \frac{r\sin C - (p-b)\cos C}{\sin C}$$

En appliquant la *transformation continue* en A, en B, en C à ces expressions de coordonnées on a immédiatement les coordonnées, par rapport à ces mêmes axes, des centres O_a, O_b, O_c des cercles ex-inscrits.

O_a : $$x = \frac{c\cos B - (p-c)}{\sin C}, \quad y = -\frac{r_a\sin C + (p-c)\cos C}{\sin C}$$

O_b : $$x = \frac{c\cos B - p}{\sin C}, \quad y = \frac{r_b\sin C - p\cos C}{\sin C}$$

O_c : $$x = \frac{c\cos B + (p-a)}{\sin C}, \quad y = \frac{r_c\sin C + (p-a)\cos C}{\sin C}$$

L'équation, en coordonnées normales, de l'ellipse qui a pour foyers deux sommets du triangle, B et C par exemple, et passe par le troisième en A, est

$$p(p-a)(b^2y^2 + c^2z^2) + bcyz\{p^2 + (p-a)^2\} + abcx(b+c)(y+z) = 0$$

Si on la transforme en A, elle se reproduit, mais si on la transforme soit en B, soit en C, on obtient l'équation de l'hyperbole qui a pour foyers B et C et passe en A

$$(p-b)(p-c)(b^2y^2 + c^2z^2) - bcyz\{(p-b)^2 + (p-c)^2\} + abcx(b-c)(y-z) = 0$$

Je vais donner quelques explications sur la façon dont j'ai obtenu les nombreuses formules auxquelles je fais souvent allusion ici, et que j'ai employées, sans les démontrer ; elles dérivent des formules connues

$$S = pr, \text{ etc.}, \quad r_a + r_b + r_c = 4R + r,$$
$$p(p-a) = r_b r_c, \quad (p-b)(p-c) = rr_a, \text{ etc.}$$

et de quelques autres que j'ai rencontrées, et qui ne l'étaient pas ou du moins dont on n'avait pas remarqué la fécondité. Je citerai par exemple les trois suivantes

$$\cos A = \frac{2R + r - r_a}{2R}$$
$$a^2 + b^2 + c^2 = 2(p^2 - r\delta)$$
$$bc + ca + ab = p^2 + r\delta$$

La première peut se démontrer ainsi.

Soient x, y, z les perpendiculaires abaissées du centre du cercle circonscrit sur les trois côtés, on a :

(1) $$x + y + z = R + r$$

C'est un théorème de Carnot dont M. J. S. Mackay a donné de nombreuses démonstrations dans son intéressant mémoire *The Triangle and its Six Scribed Circles* (Edinburgh Mathematical Society, 1883).

[redacted]

et *ce serait à désirer*, mais voici la façon de les obtenir ensemble. On a

$$r_a + r_b + r_c = 4\mathrm{R} + r = \delta$$

d'où

$$\frac{1}{b+c-a} + \frac{1}{c+a-b} + \frac{1}{a+b-c} = \frac{2\mathrm{S}}{\delta}$$

puisque $\mathrm{S} = r_a(p-a)$, etc.

De là je tire

$$\frac{a^2-(b-c)^2+b^2-(c-a)^2+c^2-(a-b)^2}{16\mathrm{S}^2} = \frac{4\mathrm{S}p}{\delta}$$

d'où

$$2\Sigma bc - \Sigma a^2 = 4\delta r$$

Mais on a identiquement

$$2\Sigma bc + \Sigma a = 4p^2$$

d'où l'on tire

$$\Sigma bc = p^2 + r\delta \quad \text{et} \quad \Sigma a^2 = 2(p^2 - r\delta)$$

Je bornerai là ce que je veux dire de ces formules, mais puisque c'est M. J. S. Mackay qui me fait l'honneur de présenter cette note à la Société Mathématique d'Edinburgh, je veux aussi ajouter quelques observations relatives à la Géométrographie qu'il vous a fait connaître, il y a quelques mois, en l'appliquant devant vous à la recherche du symbole, de la simplicité et de l'exactitude des constructions données dans l'Euclide, employé presque universellement, en Angleterre, pour l'étude des éléments de Géométrie.

Nous sommes sur le sujet, tout à fait d'accord, nous ne différons que sur des détails très peu importants au fond; seulement en dehors de l'avantage sérieux d'avoir partout identiquement les mêmes notations, je pense que celles que j'emploie, sont plus dans l'esprit de la méthode telle que je l'ai conçue, et je vais essayer brièvement de le convaincre.

Je rappelle que l'essence de la Géométrographie est spéculative, elle ne s'applique aux constructions à effectuer que—*si parva licet componere magnis*—comme la Mécanique rationnelle s'applique à l'art de l'Ingénieur. Voici mes notations

1. Faire passer le bord d'une règle par un point placé, c'est *l'opération* (R_1)

donc, *spéculativement*, faire passer le bord d'une règle par deux points placés, c'est op : $(2R_1)$

2. Tracer la droite qui suit le bord d'une règle, c'est op : (R_2)

3. Mettre une pointe d'un compas en un point placé, c'est op : (C_1)

donc, *spéculativement*, prendre avec le compas une longueur placée, c'est op : $(2C_1)$

4. Mettre une pointe en un point indéterminé d'une ligne tracée, c'est op : (C_2)

5. Tracer le cercle, c'est op : (C_3)

Monsieur Mackay supprime l'opération C_2 qu'il assimile à C_1, parceque, dit-il, quand on met la pointe en un point indéterminé d'une ligne on vise d'abord le point où l'on veut placer la pointe, c'est à dire qu'on la met réellement en un point déterminé; enfin l'on a ainsi l'avantage d'avoir plus de symétrie dans les symboles, puisque, supprimant mon symbole C_2, il appelle C_2 ce que j'appelle C_3 et que l'on a : R_1, R_2 symboles pour la droite, C_1, C_2 symboles pour le cercle.

[redacted]

Géométrographie, celle qui étudie les constructions canoniques de la règle et du compas, elle disparaîtrait avec l'adjonction des symboles nouveaux qu'amène l'emploi de l'équerre, ainsi que je l'ai développé dans un mémoire présenté au mois d'Août cette année au Congrès de l'Association Française, à Caen. Ajoutons que la question reste, en réalité, presque du domaine de la théorie, car le symbole C_2 se rencontre assez rarement dans les constructions et généralement par faibles unités dans celles où il se rencontre.

Enfin nous avons apprécié différemment dans un cas—très peu important également—la manière de compter les symboles.

Pour tracer deux droites AB, AC *passant par un même point*, je ne tiens pas compte de ce qu'elles passent par le même point et j'évalue le tracé $4R_1+2R_2$ comme s'il s'agissait de deux droites différentes AB, CD. Cela, par la raison qu'on ne peut maintenir la règle en A (une fois qu'on a tracé AB). Pour tracer AC on recommence simplement l'opération sans que la nouvelle opération profite en quoique ce soit de ce qu'on a fait pour la première. Monsieur Mackay compte $2R_1+R_2$ pour la première et seulement R_1+R_2 pour la seconde, comme si la règle avait tourné autour de A. Je m'aperçois très bien que je m'appuie sur deux ordres de raisons qui semblent se contredire ; en effet pour justifier l'emploi de C_3, je m'appuie sur l'essence spéculative de la Géométrographie et, pour justifier mon évaluation du tracé successif de deux droites qui ont un point commun, j'invoque la manœuvre pratique que l'on exécute. Cela est fort naturel cependant, parceque, quoique spéculative, la Géométrographie a en vue l'application possible de ses spéculations, elle a donc deux faces et, si elle doit rester tout à fait spéculative *lorsque cela n'a aucun inconvénient*,—c'est ainsi que pour tracer le cercle $O(b)$ immédiatement après avoir tracé le cercle $O(a)$, je ne compte que C_3, si je n'ai pas eu à déplacer la pointe fixée en O, mais seulement à ouvrir ou à fermer la branche du compas, parceque *je puis* opérer ainsi—il me semble utile de condescendre à la pratique dans le cas où j'ai à tracer deux droites successives ayant un point commun *puisque le contraire aurait un inconvénient* et que je suis, de plus près, la manœuvre du tracé.

Je ne m'étendrai pas sur l'emploi que j'ai fait de l'équerre en appliquant la Géométrographie à la Géométrie descriptive, je renvoie pour cela à mon mémoire déjà cité, présenté au Congrès de Caen. Je vais seulement définir les nouveaux symboles que j'ai adoptés en les ajoutant aux anciens.

Je suppose que l'on emploie l'équerre seulement pour tracer les parallèles et jamais pour tracer les perpendiculaires ; je ne suis point habitué à la pratique du dessin, mais je sais que cette manière d'opérer est proscrite dans toutes les épures exactes. Par exemple les instructions données aux dessinateurs des services de la Ville de Paris recommandent de ne jamais tracer de perpendiculaires avec l'équerre ; si l'on a plusieurs perpendiculaires à tracer à une même direction, on détermine la première avec la règle et le compas et les autres avec l'équerre, comme étant parallèles à celle-ci.

Nous admettons cependant que l'on se serve d'un té dont la direction donne les parallèles à la ligne de terre sur la feuille collée sur la planche, et alors, que l'on puisse mener, à l'équerre, les lignes de rappel relatives à cette ligne de terre, après avoir vérifié l'exactitude du tracé une fois pour toutes.

On n'a donc, pour l'usage de l'équerre, que les opérations nouvelles suivantes (sans compter celles où l'équerre sert comme servirait la règle seule, opérations qui conservent le symbole adopté).

—*Mettre un bord de la règle ou de l'équerre en coincidence avec une droite déjà tracé sur la figure.* Nous pourrions assimiler cette opération à faire passer le bord par deux points et compter alors $2R_1$, mais il nous semble utile—et sans aucun inconvénient—de distinguer, légèrement, cette nouvelle opération, afin de pouvoir apprécier, d'un coup d'oeil jeté sur le symbole final de la construction, le degré de fréquence dans l'emploi qu'on y a fait de l'équerre ;

[redacted]

Je veux terminer en vous donnant les résultats d'une expérience que la complaisance de M. Jung, le savant professeur de Milan, m'a permis d'exécuter et qui montre jusqu' où peut aller la simplification que la Géométrographie introduit dans une construction, simplification qui est due surtout à l'idée fondamentale de la Géométrographie, c'est à dire à la recherche *méthodique* des simplifications ; car la Géométrographie n'innove rien *en Géométrie*, elle se sert d'éléments connus, et, *à la rigueur*, toutes les simplifications auxquelles elle conduit, auraient pu théoriquement être faites sans elle ; mais on n'y songeait pas, parceque l'on manquait de criterium et parceque le mot de simplicité ne s'appliquait qu'à l'exposition théorique de la solution d'un problème et non à sa construction effectuée.

Voici l'expérience dont il s'agit. Je voulais prier un géomètre, tout à fait étranger à la notion de Géométrographie, de m'écrire en détail, *sans faire l'épure*, comment il s'y prendrait pour exécuter, *avec la règle et le compas*, le plus simplement qu'il le pourrait, selon les règles connues de la construction des expressions algébriques, une construction quelconque que je lui indiquerais au hasard. D'après cela je pourrais évaluer sa construction par le symbole géométrographique, puis reprendre, moi, le même problème en y apportant les simplifications méthodiques de la Géométrographie ; enfin évaluer ma construction et comparer les deux symboles. Je ne pouvais faire facilement la chose à Paris car les mathématiciens qui y sont de mes relations connaissent, au moins vaguement, la Géométrographie, et je n'aurai pas eu la construction dans les conditions qu'il me fallait. M. Jung avec qui j'ai le plaisir d'être en rapport, précisément à propos de Géométrographie, accepta de m'aider à faire l'expérience. Je lui envoyai alors cette construction :

Dans un triangle ABC dont les côtés sont a, b, c placer le point dont les distances aux trois côtés BC, CA, AB, (ou coordonnées normales), sont respectivement proportionelles à :

$$\frac{a^2b^2 + a^2c^2 - b^2c^2}{a}, \quad \frac{b^2c^2 + b^2a^2 - a^2c^2}{b}, \quad \frac{c^2a^2 + c^2b^2 - a^2b^2}{c}$$

c'est un point que je choisis au hasard parmi ceux qui se rencontrent fréquemment dans la Géométrie du triangle avec des coordonnées un peu complexes.

Il était convenu que le point devait être placé directement d'après ses coordonnées et non d'après des propriétés géométriques particulières à ce point, si l'on en trouvait. Monsieur Jung pria un de ses anciens élèves de faire ce que je désirais, et quelques jours après il eut la bonté de m'envoyer la rédaction même qui lui avait été remise.

Je dois dire que la construction qu'on y indique est très logique, fort bien conçue et analogue à celles que chacun de vous ferait sans doute, et que moi-même j'eusse fait il y a quelques années. On en trouverait, facilement, sans Géométrographie, de beaucoup plus simples à tracer, mais il n'y aurait aucun criterium pour permettre de l'affirmer, et cette préoccupation du tracé *réel* n'est pas dans l'esprit des Géomètres.

Je déterminai alors le symbole de cette construction que je conduisis *même assez économiquement* en évitant certaines répétitions de lignes inutiles. Puis je cherchai une construction de ce même point en appliquant les procédés géométrographiques et j'en déterminai aussi le symbole. Les résultats sont *stupéfiants* ; les voici :

Symbole de la construction qui m'a été envoyée

$$\text{op} : (81R_1 + 46R_2 + 211C_1 + 121C_3)$$

Simplicité 459 ; exactitude 292 ; 46 droites, 121 cercles.

Symbole de la construction géométrographique

$$\text{op} : (14R_1 + 7R_2 + 48C_1 + 22C_3)$$

[redacted]

surpris de cette invraisemblable différence, je pensais bien que je simplifierais d'au moins la moitié, mais de 5 fois, de 7 fois, je ne pouvais m'y attendre, d'autant plus que lorsque j'essayais moi-même d'exécuter les constructions d'application par les méthodes usuelles, pour faire la comparaison avec les méthodes géométrographiques, je les simplifiais malgré moi, pour éviter de les compliquer inutilement, et je n'arrivais pas alors à des résultats aussi éloignés dans les constructions traitées par moi avec les deux méthodes ; en m'adressant à un géomètre quelconque, j'ai évité cette cause d'incertitude, et, sans croire que *toutes* les constructions d'application—je ne parle pas des constructions fondamentales séculaires, que j'ai cependant presque *toutes* simplifiées peu ou beaucoup depuis : *mener par un point* A *une droite parallèle à une droite donnée* BC—donneraient lieu à une aussi considérable réduction, j'estime que la construction géométrographique serait *toujours* de 2 à 3 fois plus simple que la construction exécutée par les méthodes employées jusqu'ici.

PARIS, *Octobre* 1894.

NOTE ADDITIONELLE.

Si l'on convient que les longueurs doivent être comptées positivement à partir de o vers oA, on voit que AA_1 est égal à $-R_a$, l'origine des rayons étant en A. Les équations de la page 10 doivent donc être $X = \rho - R_a$, etc. Le signe du coefficient de ρ dans l'équation (1) sera simplement changé, et l'on aura les valeurs de ρ avec le signe qui leur convient.

The Nine-point Circle.

By R. F. Davis, M.A.

The nine-point circle of a triangle touches the inscribed circle.

Figure 2.

I. Let ABC be a triangle, having $\angle C$ greater than $\angle B$, D, E, F the middle points of the sides, and AX perpendicular to BC.

Then the upper segment of the nine-point circle cut off by DX contains an angle $C - B$; and conversely.

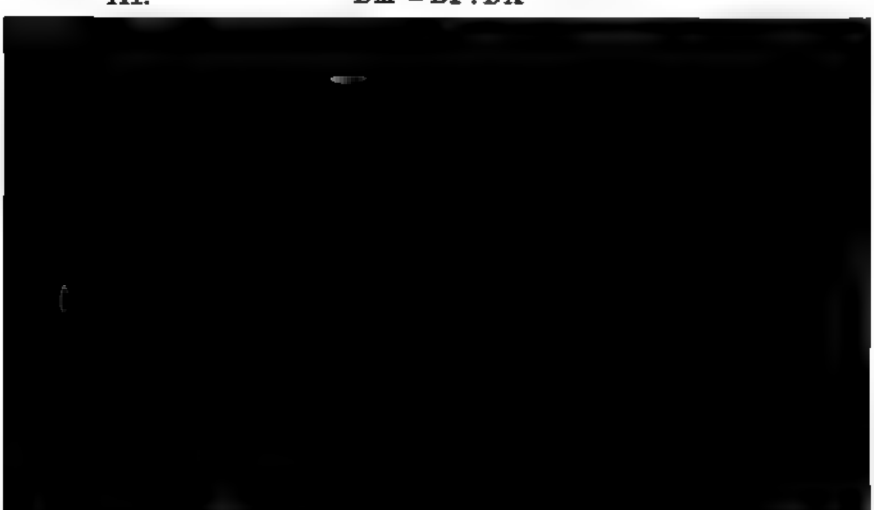
Figure 3.

II. If AP bisect the angle A and meet the base BC in P, and AC′ be taken along AB equal to AC; then PC′ touches the inscribed circle. Also $\angle BPC' = \angle C - \angle B$.

For the triangles APC, APC′ are congruent; hence the perpendiculars IM, IM′ on PC, PC′ respectively are equal.

Figure 4.

III. $DM^2 = DP \cdot DX$



For if PR, the second tangent to S from P, be drawn, and OR produced to meet S in T, since

$$OR \, . \, OT = OA^2 = OP \, . \, OQ,$$

therefore $\angle OTQ = \angle OPR$;

therefore the point T lies on Σ.

Again, drawing the tangent TU to S at T to meet PR produced in U;

$$\angle UTR = \angle URT = \angle ORP = \angle OQT;$$

therefore TU touches Σ at T.

Thus the circles S, Σ touch each other in T.

V. The application of IV. is fairly obvious. Since in Figure 4,

$$DM^2 = DP \, . \, DX \text{ (III.)},$$

the segmental circle upon DX, containing an angle

$$BPC' = C - B \text{ (II.)},$$

touches the inscribed circle (V.). But (I.) the former circle is none other than the nine-point circle.

The Brocard Points and the Brocard Angle.

By R. F Davis, M.A.

Figure 6.

I. Construction for the Brocard points.

Let ABC be a triangle. Describe a circle touching AB in A and passing through C; draw the chord AP parallel to BC. Join BP meeting this circle in Ω.

Join $A\Omega$, $C\Omega$.

Then $$\angle \Omega AB = \angle \Omega CA,$$
$$= \angle \Omega PA$$
$$= \angle \Omega BC.$$

Similarly for Ω'.

II. Characteristic property of the Brocard angle.

Draw AX, PR perpendicular to BC.

Since AP, CQ are parallel chords,
the triangles ACX, PQR are congruent by symmetry;
therefore $$AX = PR, \quad CX = QR.$$

Now $$BR = BX + CX + CR$$

[redacted]

Second Meeting, December 14th, 1894.

JOHN M'COWAN, Esq., M.A., D.Sc., President, in the Chair.

Parabolic Note: Co-Normal Points.

BY R. TUCKER, M.A.

1. If the coordinates of a point on a parabola,

$$y^2 - 4ax = 0,$$

be $(am^2, 2am)$, in which I call m the parameter, then the equations to the tangent and normal at the point are

$$x - my + am^2 = 0 \quad \ldots \quad \ldots \quad \text{(i.)}$$

and
$$mx + y - a(m^3 + 2m) = 0 \quad \ldots \quad \ldots \quad \text{(ii.)},$$

and to the chord through (m), (m') is

$$y(m + m') - 2x - 2amm' = 0 \quad \ldots \quad \ldots \quad \text{(iii.)}.$$

If we write (ii.) in the form

$$am^3 + (2a - x)m - y = 0 \quad \ldots \quad \ldots \quad \text{(iv.)},$$

we see that from a given point (x, y) we can draw three normals to the curve with the condition

$$\Sigma m = 0.$$

Let O be the point (x, y), and $P(m_1)$, $Q(m_2)$, $R(m_3)$ the corresponding points on the parabola: then I call these latter *co-normal* points, and the circle through them a *co-normal* circle.

2. We have
$$S_1 \equiv \Sigma m = 0,$$
$$S_2 \equiv \Sigma m^2 = -2\Sigma m_1 m_2,$$
$$S_3 = 3m_1 m_2 m_3 = 3\mu,$$
$$S_4 = S_2^2/2\,;$$

also
$$m_1^2 - m_2 m_3 = m_1^2 + m_1 m_2 + m_2^2 = \cdots = S_2/2.$$

3. In the case when P, Q, R are *any* three points on the curve the circle PQR is

$$x^2 + y^2 - ax[S_2 + \Sigma m_1 m_2 + 4] + ay[S_1 \, . \, \Sigma m_1 m_2 - \mu]/2 - a^2 \mu S_1 = 0 \quad \ldots \quad \text{(i.)}$$

and the tangent-circle pqr is

$$x^2 + y^2 - ax[1 + \Sigma m_1 m_2] - ay[S_1 - \mu] + a^2 \Sigma m_1 m_2 = 0, \quad \ldots \quad \text{(ii.)}$$

If the points are co-normal points, then these equations take the form

$$x^2+y^2-ax(S_2+8)/2-ay\mu/2=0, \qquad \ldots \qquad \ldots \text{(iii)}$$

$$x^2+y^2+ax(S_2-2)/2+ay\mu-a^2S_2/2=0 \ldots \qquad \ldots \text{(iv.)}.$$

4. The co-ordinates of p (§ 3) for co-normal points, are

$$(am_2m_3, \ -am_1).^*$$

5. Through P, Q, R draw parallels to

(i.) the tangents at Q, R, P;

(ii.) ,, R, P, Q;

and let P_q, Q_r, R_p; P_r, Q_p, R_q be the points where the sets (i.), (ii.) respectively meet the parabola (C.P. § 19). If PP_q, QQ_p meet in R_{rr}, and in like manner for the other pairs, then R_r is given by $a(m_1^2+m_2^2-m_1m_2), \ -am_3$. (C.P. § 24.)

Then the area of $P_pQ_qR_r$

$$= \pm\frac{a^2}{2}\begin{vmatrix} 1 & 1 & 1 \\ S_2/_2-2m_2m_3 & S_2/_2-2m_3m_1 & S_2/_2-2m_1m_2 \\ m_1 & m_2 & m_3 \end{vmatrix} = \text{area of } \triangle PQR;$$

and the equation to the circle is

$$x^2+y^2+(1-4S_2)ax/2+4\mu ay+3(S_2^2-S_2)a^2/4=0$$

we see that r_1 is given by

$$-a(2+m_1m_2),\quad -a(2m_3+\mu),$$

hence $\Delta p_1q_1r_1 = \pm\frac{1}{2}a^2 \begin{vmatrix} 1 & 1 & 1 \\ 2+m_2m_3 & 2+m_3m_1 & 2+m_1m_2 \\ \mu+2m_1 & \mu+2m_2 & \mu+2m_3 \end{vmatrix} = \Delta\mathrm{PQR}$;

and since $p_1q_1 = \mathrm{PQ}$, the triangles are congruent.

The equation to $\mathrm{A}r_1$ is $y = m_3x$,
hence $\mathrm{A}r_1$ and the normal at R make equal angles with the axis.

The circle $p_1q_1r_1$ is given by

$$x^2+y^2-(8-\mathrm{S}_2/2)ax+5a\mu y/2+3a^2(\mu^2-2\mathrm{S}_2+8)/2=0\,;$$

and the N.P. circle is given by

$$x^2+y^2+(2-\mathrm{S}_2/4)ax+7a\mu y/4-a^2(2\mathrm{S}_2-3\mu^2)/4=0.$$

7. The equations to PP_p, QQ_q (§ 6) are

$$m_1x-m_2m_3y=am_1(m_1^2-2m_2m_3),$$
$$m_2x-m_3m_1y=am_2(m_2^2-2m_3m_1),$$

hence they intersect in r_2, on the ordinate of R, given by

$$am_3^2,\quad -am_1m_2/m_3.$$

Similarly for the analogous points p_2, q_2.

Hence $$\Delta p_2q_2r_2=\tfrac{1}{2}\,\Delta\mathrm{PQR}.$$

The circle $p_2q_2r_2$ is given by

$$x^2+y^2-(\mathrm{S}_2+1)ax+(\mu+\mathrm{S}_2^2/4\mu)ay+a^2(4\mathrm{S}_2+\mathrm{S}_2^2)/4=0.\quad\ldots\quad\text{(i.)}$$

The points $p_2q_2r_2$, lie on the rectangular hyperbola

$$xy=-\mu a^2\ \ldots\quad\ldots\quad\ldots\quad\ldots\ \text{(ii.)}$$

which cuts (i.) again in $(a,\ -a\mu)$.

The equation to the perpendicular from r_2 on p_2q_2 is

$$m_3y+m_1m_2x=am_1m_2(m_3^2-1),$$

hence the orthocentre of $p_2q_2r_2$ is $(-a,\ a\mu)$.

This point is on (ii) and *coincides with the orthocentre* of pqr (C.P. § 13).

The N.P. circle of $p_2q_2r_2$ is the co-normal circle

$$x^2+y^2+ax(1-S_1)/2+ay(S_2^2-4\mu^2)/8\mu=0.$$

The radical axis of this circle and of PQR is

$$36\mu^2x+S_3^2y=0.$$

The tangent from the focus to (i.) is $aS_2/2$.

The equation to pp_2 is

$$S_2x-6\mu y=2a(m_1^4-m_1\mu+3m_2^2m_3^2),$$

hence pp_2, qq_2, rr_2 are parallel.

Also the equation to p_2q_2 is

$$m_1m_2y-m_3x=-a(m_1^2+m_2^2)m_3,$$

i.e., the line is parallel to Rr.

The points p, q, r lie on the hyperbola (ii.): hence we see otherwise that the orthocentres of pqr, $p_2q_2r_2$ coincide, and that the two circles cut the Latus Rectum in the same point $(a, -a\mu)$, the join of which with the common orthocentre is a diameter of the hyperbola.

[redacted]

The orthocentre of LMN is

$$a(S_2-4)/2, \quad -2a\mu.$$

If n is the midpoint of LM, it is given by

$$(-am_1m_2, \ -am_3)$$

and therefore it and the analogous points l, m, lie on the rectangular hyperbola $$xy=\mu a^2. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{(i.)}$$

From the above we see that pl, qm, rn are diameters of the parabola, and lm, pq ; mn, qr ; nl, rp ; intersect on the tangent at the vertex and are isoclinals to it.

The equation to the circle lmn is

$$x^2+y^2+ax(2-S_2)/2+a\mu y-a^2S_2/2=0\,;$$

and it is therefore § 3 (iv.) equal to the circle pqr.

The perpendiculars from l, m, n on QR, RP, PQ respectively meet in $(2a, a\mu/2)$, which is on (i.); and the perpendiculars from P, Q, R on mn, nl, lm meet in $[a(S_2-4)/2, \ -a\mu]$, *i.e.* O′ of C.P. § 15.

10. If the join of P to the midpoint of QR cuts the parabola in p_3, the parameter of this point is $-S_2/3m_1$, hence the corresponding tangent circle of the triangle $p_3q_3r_3$ is given by

$$x^2+y^2-ax-ayS_2^2(9+2S_2)/54\mu=0.$$

The vertices of this tangent triangle are

$$(aS_2^2/9m_1m_2, \ am_3S_2/3m_1m_2)$$

so that its centroid is $\quad (0, \ -S_2^2/18\mu)$;

and its orthocentre $\quad (-a, \ 7aS_2^3/54\mu)$.

11. Through P, Q, R draw lines parallel to QR, RP, PQ respectively, these lines meet the parabola in the co-normal points, whose parameters are $-2m_1$, $-2m_2$, $-2m_3$; and the lines cut one another in

$$(-2am_2m_3, \ -4am_1), \ (-2am_3m_1, \ -4am_2), \ (-2am_2m_3, \ -4am_3).$$

Take the images of these points in the vertex, viz $(2am_1m_3, 4am_1)$, etc., and we find its circumcircle to be given by

$$x^2+y^2-(8-S_2)ax-a\mu y-8S_2a^2=0,$$

the centre of which is the orthocentre of PQR (C.P. § 13.)

12. The lines QR, AP cut in $p_a\,(-am_2m_3/2,\ -am_2m_3/m_1)$, RP, AQ in q_a, and PQ, AR in r_a; hence $p_aq_ar_a$, which is the *central* triangle of the quadrilateral APQR,

$$=\tfrac{1}{4}\triangle PQR.$$

The circle $p_aq_ar_a$ has for its equation

$$x^2+y^2+2ax+ay(S_2^2/\mu^2)/4\mu+a^2S_2/2=0.$$

The equation to p_aq_a is

$$m_1m_2y+2m_3x=+a\mu.$$

13. If (cf. C.P. §29) we draw lines from P, Q, R through the point, $x=ka$ on the axis to cut the curve in T_1', T_2', T_3', then as T_1' is given by $(-k/m_1)$ the equation to the tangent-circle for $T_1'T_2'T_3'$ will differ from that to the tangent-circle for $T_1T_2T_3$ only in the sign of k, *i.e.*, it will be

$$x^2+y^2-ax-ay(k\,.\,S_2+2k^2)/2\mu=0.$$

[redacted]

The equations to the co-normal circles are

$$x^2 + y^2 - ax(3S_2 + 8)/2 \mp ayk/2 = 0,$$

where $$k \equiv m_2 - m_1 \,.\, m_3 - m_1 \,.\, m_1 - m_2.$$

17. The median of PQR which passes through P cuts the parabola in the point whose parameter is $(-S_2/3m_1)$, hence the corresponding tangent-circle has for its equation

$$x^2 + y^2 - ax - ayS_2^2(9 + 2S_2)/54\mu = 0.$$

18. If in § 12 q_a, r_a are outside the curve, then the midpoints of QR, AP, q_ar_a are given by

$$a(m_2^2 + m_3^2)/2,\ -am_1\ ;\ \ am_1^2/2,\ am_1\ ;\ \ am_1^2/4,\ -am_1(m_2^2 + m_3^2)/2m_2m_3\ :$$

hence the *central* axis of APQR is

$$-2m_2m_3y + 4m_1x = m_1S_2a.$$

19. The poles of the co-normal chords are

$$-a(m_1^2 + 2),\ -2a/m_1\ ;\ \ -a(m_2^2 + 2),\ -2a/m_2\ ;\ \ -a(m_3^2 + 2),\ -2a/m_3.$$

These poles lie upon the line

$$\mu y - 2x = a(S_2 + 4). \qquad \text{(cf. C.P. § 17.)}$$

The diameters through the poles meet the curve in

$$a/m_1^2,\ -2a/m_1\ ;\ \text{etc.}\ ;$$

hence the circle through the vertices of these diameters is

$$x^2 + y^2 - ax(S_2^2 + 4\mu^2)/\mu^2 + ay/2\mu + a^2S_2/2\mu^2 = 0\ ;$$

and the corresponding tangent-circle is

$$x^2 + y^2 - ax - ay(S_2 + 2)/2\mu = 0.$$

The sides of this last triangle are

$$m_1y - 2m_2m_3x = 2a, \quad \text{etc.},$$

∴ the perpendiculars are

$$m_1^2x + 2\mu y = a(1 - 4m_2m_3), \quad \text{etc.},$$

whence the orthocentre is

$$[-4a,\ a(1 + 2S_2)/2\mu].$$

Geometrical Problem.

By G. E. Crawford, M. A.

Figure 24.

Let OQ, OR be two straight lines meeting at O, and P any point. Required to draw through P a straight line cutting off a given area OAB from the two straight lines.

Draw PD parallel to OR cutting OQ in D.

Construct a $\triangle$OPC equal to the given area, and such that OP is one of its sides, and that another of its sides, OC, lies along OQ.

Take OE a mean proportional to OC, OD.

Draw OF perpendicular to OC and equal to half of it.

Join EF, and cut off FG = OF.

Take OA = EG. Then PAB is the required straight line.

Proof:

Sqs. on OE, OF = sq. on EF

= sqs. on EG, GF, + 2 rect EG . GF

$\therefore$ sq. on OE = sq. on EG + 2 rect. EG . GF

= sq. on OA + rect. OA . OC (since OC = 2GF)

$\therefore$ rect. OC . OD = sq. on OA + rect. OA . OC

[redacted]

Third Meeting, January 11th, 1895.

WM. PEDDIE, Esq., D.Sc., F.R.S.E., Vice-President, in the Chair.

Properties connected with the Angular Bisectors of a Triangle.

By J. S. MACKAY, M.A., LL.D.

NOTATION.

When points and lines are not specifically designated in the course of the following pages it will be understood that the notation for them is that recommended in the *Proceedings of the Edinburgh Mathematical Society*, Vol. I. pp. 6–11 (1894). It may be convenient to repeat all that is necessary for the present purpose.

A′ B′ C′ = mid points of the sides BC CA AB

D E F = points of contact of sides with incircle

D_1 E_1 F_1 = " " " " " " first excircle.

And so on.

H = orthocentre of ABC

I = incentre of ABC

I_1 I_2 I_3 = 1st 2nd 3rd excentres of ABC

L M N = feet of interior angular bisectors of ABC

L′ M′ N′ = " " exterior " " " "

O = circumcentre of ABC

U U′ = ends of that diameter of the circumcircle which is perpendicular to BC. U is on the opposite side of BC from A.

Similarly for V V′ and W W′.

X Y Z = feet of the perpendiculars from A B C.

The various points

K K′, P P′, Q Q′, S S′, T T′

are defined as they occur.

α	β	γ	$=$ AI	BI	CI	
α_1	β_1	γ_1	$=$ AI$_1$	BI$_1$	CI$_1$	
α_2	β_2	γ_2	$=$ AI$_2$	BI$_2$	CI$_2$	
α_3	β_3	γ_3	$=$ AI$_3$	BI$_3$	CI$_3$	
$\alpha_1-\alpha$	$\beta_2-\beta$	$\gamma_3-\gamma$	$=$ I$_1$I	I$_2$I	I$_3$I	
$\alpha_2+\alpha_3$	$\beta_3+\beta_1$	$\gamma_1+\gamma_2$	$=$ I$_2$I$_3$	I$_3$I$_1$	I$_1$I$_2$	
h_1	h_2	h_3	$=$ the perpendiculars AX BY CZ			
l_1	l_2	l_3	$=$ the interior angular bisectors of A B C			
λ_1	λ_2	λ_3	$=$ „ exterior „ „ „ „ „ „			
	r		$=$ radius of the incircle			
r_1	r_2	r_3	$=$ radii of the 1st 2nd 3rd excircles			
	R		$=$ radius of the circumcircle			
	s		$=$ semiperimeter of ABC			
s_1	s_2	s_3	$= s-a$	$s-b$	$s-c$	
u_1	v_1	w_1	$=$ BL	CM	AN	
u_1'	v_1'	w_1'	$=$ BL′	CM′	AN′	
u_2	v_2	w_2	$=$ CL	AM	BN	
u_2'	v_2'	w_2'	$=$ CL′	AM′	BN′	

Then $\quad BP = B_1P \qquad BP' = B_2P'$

and $\quad CB_1 = AC - AB \qquad CB_2 = AC + AB.$

Now since $P\ P'$ are the mid points of $BB_1\ BB_2$ and A' the mid point of BC,

therefore
$$A'P = \tfrac{1}{2}CB_1 \qquad A'P' = \tfrac{1}{2}CB_2$$
$$= \tfrac{1}{2}(AC - AB) \qquad = \tfrac{1}{2}(AC + AB)$$

Similarly if $Q\ Q'$ be the feet of the perpendiculars from C on $AL\ AL'$,
$$A'Q = \tfrac{1}{2}(AC - AB) \qquad A'Q' = \tfrac{1}{2}(AC + AB).$$

It is not easy to assign authorities to the properties given in the following pages. Some of these properties occur incidentally in the solutions of problems on the construction of triangles, and are there spoken of, or assumed without being spoken of, as well known theorems. A large collection of them will be found in four articles entitled "Useful Propositions in Geometry" by M. A. Harrison, which appeared in Leybourn's *Mathematical Repository*, old series, I. 283-5, 367-9, II. 23-5, 234-7 (1799-1801). In these articles no mention is made of properties connected with the bisector of the exterior vertical angle.

It has been conjectured that "M. A. Harrison" is a pseudonym, adopted either by J. H. Swale or John Lowry.

(1) $\quad \angle ABB_1 = AB_1B = BAL' = B_2AL' = AP'A' = AQ'A'$
$$= \tfrac{1}{2}(B + C)$$
$$\angle CBB_1 = BL'A = \tfrac{1}{2}(B - C)$$

(2) $A'PP'$ is a straight line parallel to AC. Hence $P\ P'$ are situated on $C'A'$ one of the sides of the triangle $A'B'C'$, which is complementary to ABC.

Similarly, if from B perpendiculars be drawn to the bisectors of the interior and exterior angles at C, the feet of these perpendiculars will also be situated* on $C'A'$.

(3) If perpendiculars be drawn from each vertex of a triangle to the interior and the exterior bisectors of the angles at the other vertices, the twelve points of intersection thus obtained will range, four

* Arthur Lascases in the *Nouvelles Annales*, XVIII. 171 (1859).

and four, on three straight lines, which by their mutual intersections will form the triangle complementary to the given triangle. *

The proof of this is obvious enough from what precedes; but the following demonstration will be found interesting.

FIGURE 8.

Let ABC be a triangle, $I\ I_1\ I_2\ I_3$ the incentre and the ex-centres.

The four lines $I_2B\ I_3I_1\ I_3C\ I_1I_2$ are the interior and the exterior bisectors of the angles B and C. Now these four lines, taken three and three, form the four triangles

$$I_3I_1C \quad I_2IC \quad I_2I_1B \quad I_3IB$$

Hence, by a theorem due to Wallace,† the circumcircles of these four triangles all pass through the same point A; and by one of Steiner's theorems ‡ the feet of the perpendiculars let fall from A on the four straight lines are collinear.

Let $A_1\ A_2\ A_3\ A_4$ be the feet of the perpendiculars.
Then AA_1BA_2 is a rectangle;
therefore A_1A_2 passes through C' the mid point of AB.
Similarly A_3A_4 „ „ B' „ „ „ „ AC;
therefore the straight line $A_1A_2A_3A_4$ bisects AB and AC.

(4) $A_3A_4 = b, \quad A_1A_2 = c, \quad A_2A_4 = s$

A_1A [illegible] s, [illegible], [illegible] s

[redacted]

The circles AI_2 AI_3 touch each other at A, and have AI_1 for common tangent; they also touch the circle $I_2 I_3$ the former at I_2 and the latter at I_3.

(7) The radical axis of the circles

$$\begin{array}{lllll} AB & AI & AI_2 & \text{is} & AA_1 \\ AB & AI_3 & AI_1 & ,, & AA_2 \\ AC & AI & AI_3 & ,, & AA_3 \\ AC & AI_1 & AI_2 & ,, & AA_4 \end{array}$$

(8)
$$AP : AL = \tfrac{1}{2}(AC + AB) : AC$$
$$AQ : AL = \tfrac{1}{2}(AC + AB) : AB$$

FIGURE 9.

Since PA' is parallel to AC,

therefore triangles ACL $PA'L$ are similar;

therefore $$AL : PL = AC : PA'$$
$$= AC : \tfrac{1}{2}(AC - AB)$$

therefore $$AL - PL : AL = \tfrac{1}{2}(AC + AB) : AC$$

(9)
$$AP' : AL' = \tfrac{1}{2}(AC - AB) : AC$$
$$AQ' : AL' = \tfrac{1}{2}(AC - AB) : AB$$

(10)
$$PQ : AL = AC^2 - AB^2 : 2AC \cdot AB$$
$$P'Q' : AL' = AC^2 - AB^2 : 2AC \cdot AB$$

FIGURE 9.

Since triangles ACL $PA'L$ are similar

therefore $$PL : AL = PA' : AC$$
$$= \tfrac{1}{2}(AC - AB) : AC$$

Since triangles ABL $QA'L$ are similar

therefore $$QL : AL = QA' : AB$$
$$= \tfrac{1}{2}(AC - AB) : AB$$

therefore $$\frac{PL+QL}{AL} = \frac{AC-AB}{2AC} + \frac{AC-AB}{2AB}$$

$$= \frac{AC \cdot AB - AB^2}{2AC \cdot AB} + \frac{AC^2 - AC \cdot AB}{2AC \cdot AB}$$

$$= \frac{AC^2 - AB^2}{2AC \cdot AB}$$

(11)* $$ABC = AQ \cdot BP = AP \cdot CQ$$
$$= AQ' \cdot BP' = AP' \cdot CQ'$$

For triangles AXL, BPL are similar

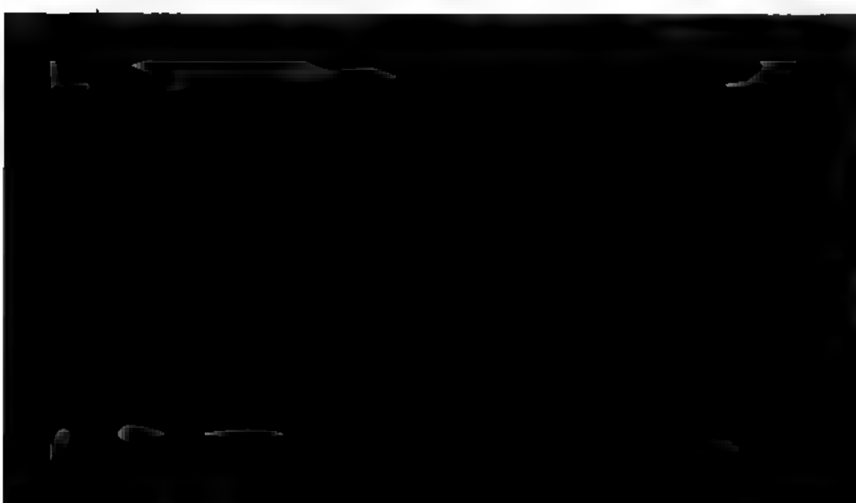
therefore $$AX : BP = AL : BL$$
$$= QL : A'L$$
$$= AQ : BA'$$

therefore $$AQ \cdot BP = AX \cdot BA'$$
$$= ABC$$

The last two expressions for ABC may be derived from the first two, since

$$AP = BP' \quad AQ = CQ' \quad BP = AP' \quad CQ = AQ'.$$

(12) The values of the following angles may be registered for



In triangles BD_1P CQD_1

$$PBD_1 = \tfrac{1}{2}(B - C), \quad BD_1P = \tfrac{1}{2}C, \quad D_1PB = 180^\circ - \tfrac{1}{2}B$$
$$= D_1CQ \qquad = CQD_1 \qquad = QD_1C$$

In triangles BD_2P' $CQ'D_2$

$$P'BD_2 = \tfrac{1}{2}A + B, \quad BD_2P' = \tfrac{1}{2}C, \quad D_2P'B = 90^\circ - \tfrac{1}{2}B$$
$$= D_2CQ' \qquad = CQ'D_2 \qquad = Q'D_2C$$

In triangles BD_3P' $CQ'D_3$

$$P'BD_3 = \tfrac{1}{2}A + C, \quad BD_3P' = 90^\circ - \tfrac{1}{2}C, \quad D_3P'B = \tfrac{1}{2}B$$
$$= D_3CQ' \qquad = CQ'D_3 \qquad = Q'D_3C$$

$$AEP = AFP = AQE_1 = AQF_1 = 90^\circ + \tfrac{1}{2}C$$
$$APE = APF = AE_1Q = AF_1Q = \tfrac{1}{2}B$$

$$AE_2P' = AF_2P' = AQ'E_3 = AQ'F_3 = \tfrac{1}{2}C$$
$$AP'E_2 = AP'F_2 = AE_3Q' = AF_3Q' = \tfrac{1}{2}B$$

(13)* $$AP \cdot AQ = BP' \cdot CQ' = s\,s_1$$
$$BP \cdot CQ = AP' \cdot AQ' = s_2 s_3$$

The similar triangles AEP AQE_1 give

$$AE : AP = AQ : AE_1$$

therefore $$AP \cdot AQ = AE \cdot AE_1$$
$$= s s_1 \quad ;$$

and $$AP = BP' \quad AQ = CQ'.$$

The other equalities may be derived from the similar triangles BDP CQD, and the fact that

$$BP = AP' \quad CQ = AQ'.$$

(14) $$AP \cdot AQ \cdot BP \cdot CQ = AP' \cdot AQ' \cdot BP' \cdot CQ'$$
$$= \Delta^2$$

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* (13) Half of this is given in Hind's *Trigonometry*, 4th ed., p. 304 (1841).

(15) Let D, D_1, D_2, D_3 be the points where the incircles and the excircles touch BC.

FIGURE 9.

It is known that

$$A'D = A'D_1 = \tfrac{1}{2}(AC - AB), \quad A'D_2 = A'D_3 = \tfrac{1}{2}(AC + AB);$$

hence D D_1 P Q lie on a circle with centre A'

and D_2 D_3 P' Q' ,, ,, ,, ,, ,, ,, ,,

(16) The incircle and first excircle of ABC cut the circle DPD_1Q orthogonally, and the second and third excircles cut $D_2Q'P'D_3$ orthogonally.

For $D\,D_1$ is perpendicular to $I\,D$ and I_1D_1;
and D_2D_3 ,, ,, I_2D_2 ,, I_3D_3.

(17) $$I\,P \cdot I\,Q = r^2, \quad I_1P \cdot I_1Q = r_1^2$$
$$I_2P' \cdot I_2Q' = r_2^2, \quad I_3P' \cdot I_3Q' = r_3^2$$

(18) If I I_1 be considered as one pair of a system of coaxal circles, then P Q are the limiting points of the system; and P' Q' are the limiting points of the coaxal system of which I_2 I_3 form one pair.

[redacted]

(20) Triangles XPQ, XP′Q′ are inversely similar* to ABC.

FIGURE 9.

Since A P B X are concyclic

therefore $\angle APX = \angle ABX$

therefore $\angle XPQ = \angle ABC$

Since A C Q X are concyclic

therefore $\angle AQX = \angle ACX$

therefore triangle XPQ is similar to ABC

In like manner XP′Q′ is similar to ABC

(21) The directly similar triangles XPQ XP′Q′ have their homologous sides mutually perpendicular.

(22) The incentre and the excentres of triangles XPQ XP′Q′ are situated on BX and AX.

Since A P B X are concyclic

therefore $\angle BXP = \angle BAP = \frac{1}{2}A$

Since A C Q X are concyclic

therefore $\angle CXQ = \angle CAQ = \frac{1}{2}A$

therefore BX bisects $\angle PXQ$

therefore BX contains the incentre and one excentre of XPQ.

Now AX is perpendicular to BX

therefore AX contains the other excentres.

In like manner it may be proved that AX contains the incentre and one excentre of triangle XP′Q′, and that BX contains the other excentres.

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* W. H. Levy in the *Lady's and Gentleman's Diary* for 1856, p. 49. The first part of the theorem, however, is given in Leybourn's *Mathematical Repository*, old series, II. 25 (1801).

(23) To determine the incentre and the excentres of the triangles XPQ XP′Q′.

Since AX ID are parallel

therefore $AL : IL = XL : DL.$

But in the similar triangles ABC XPQ
AL and XL are homologous lines ;
therefore IL and DL are homologous lines,
and I D homologous points ;
therefore D is the incentre of XPQ.

Since $\angle DPD_1$ is right, D_1 is the first excentre.

The other excentres are the points where DP and DQ intersect AX.

In like manner it may be proved that D_3 and D_2 are the third and second excentres of triangle XP′Q′ and that the incentre and first excentre are the points where D_2Q' and D_3P' intersect AX.

(24) The circumcircles* of XPQ XP′Q′ pass through A′.

Since $\angle A'XQ = \angle CAQ$

$= \angle A'PQ$

because A′P and CA are parallel ;

[redacted]

(27) The circle XPQ cuts orthogonally the system of circles $I\ I_1$; and the circle XP′Q′ cuts orthogonally the system $I_2\ I_3$.

For the circle XPQ passes through the limiting points P Q of the system $I\ I_1$ and has its centre on the radical axis of the same system.

(28) The centres of the circles XPQ and XP′Q′ and the nine-point centre of triangle ABC are collinear.

For they are situated on the straight line which bisects A′X perpendicularly.

(29) The sum of the areas of the circles XPQ XP′Q′ is equal to the area of the circle ABC.

For the areas of circles are proportional to the squares of their diameters
and $$AU'^2 + AU^2 = UU'^2.$$

(30) $$XPQ + XP'Q' = ABC.^*$$

(31)† $$ABC : XPQ = UU' : U'K$$
$$ABC : XP'Q' = UU' : UK$$

For $$ABC : XPQ = UU'^2 : AU'$$
$$= UU' : U'K$$

For another proof see §4, (11).

(32) $$XP \quad XP' \quad XQ \quad XQ'$$
are respectively parallel to
$$CU \quad CU' \quad BU \quad BU'$$

For $$\angle XPQ = \angle ABC$$
$$= \angle AUC.$$

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* W. H. Levy in the *Lady's and Gentleman's Diary* for 1855, p. 71.

† Parts of (28), (29), (30), (31), are found in Leybourn's *Mathematical Repository*, old series, II. 236, 25 (1801).

(33) The triangles

$$A'BP \quad A'CQ \quad A'BP' \quad A'CQ'$$

are respectively similar to

$$QAX \quad PAX \quad Q'AX \quad P'AX.$$

For A'P is parallel to CA ;

therefore $\angle BA'P = C$

$= \angle AQX$;

and $\angle BPA' = 90^\circ + \frac{1}{2}A$

$= \angle AXQ.$

Or it may be proved that

$$\angle A'BP = \angle QAX$$

since the sides of the one are perpendicular to the sides of the other.

(34) The triangles

$$A'UP \quad A'UQ \quad A'U'P' \quad A'U'Q'$$

are respectively similar to

[redacted]

FIGURE 10.

The points B D P I are concyclic;
therefore $\angle$ BDP is the supplement of $\angle$ BIP.
Because the isosceles triangles CDE, UBI have

$$\angle C = \angle U$$

therefore $\angle CDE = \angle BIU$;
therefore $\angle$ BDP is the supplement of $\angle$ CDE;
therefore DP coincides with DE.

(36) The following quintets of points are concyclic:

B	D	I	F	P :	C	D	I	E	Q
B	D_1	I_1	F_1	P ;	C	D_1	I_1	E_1	Q
B	D_2	I_2	F_2	P′ ;	C	D_2	I_2	E_2	Q′
B	D_3	I_3	F_3	P′ ;	C	D_3	I_3	E_3	Q′

the diameters of the various circles being

BI	BI_1	BI_2	BI_3
CI	CI_1	CI_2	CI_3

(37) Since C L B L′ form a harmonic range,
and CQ BP AL′ are parallel,
therefore Q L P A form a harmonic range.*

FIGURE 9.

Similarly for Q′ A P′ L′.

(38) If at L a perpendicular be drawn to AL meeting AB AC at P Q, then AP or AQ is a harmonic mean† between AB AC.

* Fuhrmann's *Synthetische Beweise planimetrischer Sätze*, pp. 58-9 (1890).
† Rev. R. Townsend in *Mathematical Questions from the Educational Times*, XIV. 76 (1870).

FIGURE 11.

If B_1 be the image of B in AL,
then AL bisects $\angle BLB_1$;
therefore LQ bisects $\angle CLB_1$;
therefore $L(BAB_1Q)$ is a harmonic pencil.
Now this pencil is cut by the transversal AC;
therefore A B_1 Q C form a harmonic range
and AQ is the harmonic mean between AB_1 AC.

Similarly for L'.

(39) $$AU : IU = AB + AC : BC$$

FIGURE 12.

Draw IP IQ parallel to AB AC.

The quadrilaterals $ABUC$ $IPUQ$ are similar:
therefore $$AU : IU = AB + AC : IP + IQ.$$

Now $$\angle UBL = \angle UAC = \angle UAB = \angle UIP,$$

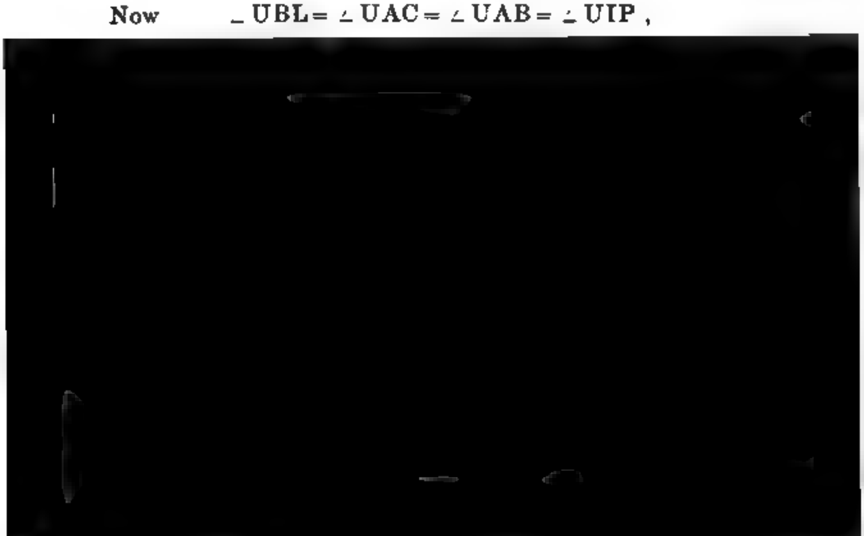

FIGURE 13.

The triangle PUU' is isosceles:

therefore $$PA'^2 = PU^2 - UA' \cdot A'U'$$
$$= PU^2 - A'B^2;$$
and $$PK^2 = PU^2 - UK \cdot KU'$$
$$= PU^2 - AK^2.$$

Now $$AB^2 + AC^2 = 2A'B^2 + 2A'A^2$$
$$= 2A'B^2 + 2AK^2 + 2A'K^2$$
$$= 2A'B^2 + 2AK^2 + 2PA'^2 + 2PK^2$$
$$- 4PU^2$$

§ 2.

If from the mid point of the base of a triangle a perpendicular be drawn to the bisector of the interior or exterior vertical angle, this perpendicular will cut off from the sides segments equal to half the sum* or half the difference of the sides.

FIGURE 14.

Let the perpendiculars from A' to AU, AU' meet AC at S S', and AB at T T'.

Draw BB_1 BB_2 parallel to the perpendiculars.

Because A' is the mid point of BC,

therefore S " " " " " B_1C.

Now $$B_1C = AC - AB;$$
therefore $$CS = \tfrac{1}{2}(AC - AB);$$
therefore $$AS = \tfrac{1}{2}(AC + AB).$$

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* Part of this is found in Leybourn's *Mathematical Repository*, old series, I. 284 (1799).

Similarly $\qquad BT = AT' = AS' = \frac{1}{2}(AC - AB)$

and $\qquad AT = BT' = CS' = \frac{1}{2}(AC + AB)$

(1) $\qquad \angle ATS = \angle AST = \angle ABB_1 = \frac{1}{2}(B + C)$

$\qquad \angle BA'T = \angle CA'S = \angle CBB_1 = \frac{1}{2}(B - C)$

(2) $\qquad AS^2 + CS^2 = AT^2 + BT^2 = \frac{1}{2}(b^2 + c^2)$

(3) $\qquad AS^2 - CS^2 = AT^2 - BT^2 = bc$

(4) $\qquad AS \cdot CS = AT \cdot BT = \frac{1}{4}(AC^2 - AB^2)$

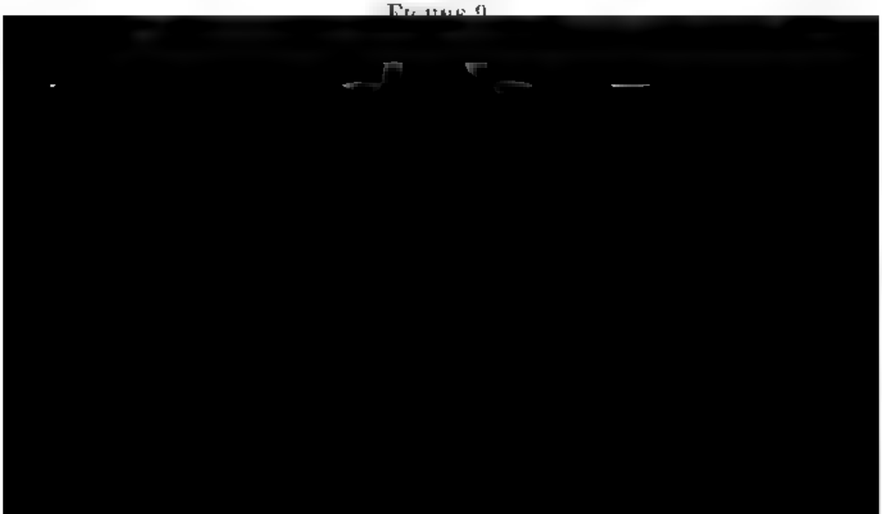
$\qquad\qquad = \frac{1}{4}(CX^2 - BX^2) = A'B \cdot A'X$

(5) $\qquad AS : CS = AT : BT = b + c : b - c$

(6) $\qquad SS' = AB = c, \quad TT' = AC = b$

Instead of drawing perpendiculars to the two bisectors of the vertical angle either from the ends of the two sides or from the mid point of the base, if perpendiculars be drawn to the sides from certain points in the two bisectors of the vertical angle, values will be obtained for half the sum and half the difference of the sides.

Figure 9



Similarly $\quad AS' = AT' \quad CS' = BT'$.

Now
$$\tfrac{1}{2}(AC + AB) = \tfrac{1}{2}\{(AS + CS) + (AT - BT)\}$$
$$= \tfrac{1}{2}\ (AS + AT) = AS = AT \quad ;$$

and also
$$= \tfrac{1}{2}\{(CS' + AS') + (BT' - AT')\}$$
$$= \tfrac{1}{2}\ (CS' + BT') = CS' = BT' \quad .$$

(8) $\qquad CS = BT = AS' = AT' = \tfrac{1}{2}(AC - AB)$

For
$$\tfrac{1}{2}(AC - AB) = \tfrac{1}{2}\{(AS + CS) - (AT - BT)\}$$
$$= \tfrac{1}{2}\ (CS + BT) = CS = BT \quad ;$$

and also
$$= \tfrac{1}{2}\{(CS' + AS') - (BT' - AT')\}$$
$$= \tfrac{1}{2}\ (AS' + AT') = AS' = AT' \quad .$$

(9)
$$\angle U'BC = U'CB = U'UB = U'UC$$
$$= U'AS' = U'AT' = AUS = AUT$$
$$= \tfrac{1}{2}(B + C)$$

(10) $\quad \angle U'BA = U'CA = U'UA = BUT = CUS$
$$= \tfrac{1}{2}(B - C)$$

For half the sum of two magnitudes increased by half their difference gives the greater.

(11) If $\ BP\ CQ\ $ be drawn perpendicular to $\ AU$,
$$\angle ABP = 90^\circ - \tfrac{1}{2}A = \tfrac{1}{2}(B + C) \ .$$

But $\qquad \angle AUS \qquad = \tfrac{1}{2}(B + C)$;

therefore $\ BP\ US\ $ intersect* on the circle $\ ABC$.

Similarly $\ CQ\ UT\ $ " " " " " .

A like statement holds good for $\ BP'\ U'S'$, and for $\ CQ'\ U'T'$.

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* Leybourn's *Mathematical Repository*, old series, I. 285 (1799).

(12)* If $\left.\begin{array}{l} BP\ \ US \\ CQ\ \ UT \\ BP'\ \ U'S' \\ CQ'\ \ U'T' \end{array}\right\}$ meet on the circumcircle at $\left\{\begin{array}{l} B_2 \\ C_2 \\ B_2' \\ C_2' \end{array}\right.$

$$\begin{aligned} 4US \cdot SB_2 &= 4UT \cdot TC_2 \\ &= 4U'S' \cdot S'B_2' = 4U'T' \cdot T'C_2' = AC^2 - AB^2 \end{aligned}$$

For $$\begin{aligned} US \cdot SB_2 &= AS \cdot SC \\ &= \tfrac{1}{2}(AC + AB) \cdot \tfrac{1}{2}(AC - AB) \end{aligned}$$

(13) The incentre I of ABC is situated on AU. If with centre U and radius UI a circle be described, it will pass through B and C and will cut AC AB again at B_1 C_1 such that $B_1S = CS \quad C_1T = BT$.

Hence $$B_1C = BC_1 = AC - AB \ ;$$

and B P B_1 are collinear, and so are C Q C_1 .

(14) $$\begin{aligned} U'B_2 &= UB_2' = AB_1 = AB \\ U'C_2 &= UC_2' = AC_1 = AC \end{aligned}$$

(15) B_2 B_2' are symmetrically situated with respect to O, and so are C_2 C_2'

[redacted]

(18) All the four circles are equal to each other, and their diameters are equal to BC.

The first two cut the circles $\mathrm{I}\ \mathrm{I}_1$ orthogonally,
and the second „ „ „ „ $\mathrm{I}_2\ \mathrm{I}_3$ „ .

Compare § 1, (16).

§ 3.

To find values for the rectangles contained by various segments of the base BC.

FIGURE 9.

The values of the segments here given will be found useful in the verification of properties (1)–(12).

$$\mathrm{BX} = \frac{a^2 - b^2 + c^2}{2a} \quad \mathrm{CX} = \frac{a^2 + b^2 - c^2}{2a} \quad \mathrm{A'X} = \frac{b^2 - c^2}{2a}$$

$$\mathrm{A'D} = \mathrm{A'D}_1 = \frac{b - c}{2}$$

$$\mathrm{A'D}_2 = \mathrm{A'D}_3 = \frac{b + c}{2}$$

$$\mathrm{BL} = \frac{ca}{b+c} \qquad \mathrm{BL'} = \frac{ca}{b-c}$$

$$\mathrm{CL} = \frac{ab}{b+c} \qquad \mathrm{CL'} = \frac{ab}{b-c}$$

$$\mathrm{A'L} = \frac{a(b-c)}{2(b+c)} \qquad \mathrm{A'L'} = \frac{a(b+c)}{2(b-c)}$$

$$\mathrm{LD} = \frac{s_1(b-c)}{b+c} \qquad \mathrm{LD}_1 = \frac{s\,(b-c)}{b+c}$$

$$\mathrm{L'D}_2 = \frac{s_3(b+c)}{b-c} \qquad \mathrm{L'D}_3 = \frac{s_2(b+c)}{b-c}$$

$$\mathrm{DX} = \frac{s_1(b-c)}{a} \qquad \mathrm{D}_1\mathrm{X} = \frac{s\,(b-c)}{a}$$

$$\mathrm{D}_2\mathrm{X} = \frac{s_3(b+c)}{a} \qquad \mathrm{D}_3\mathrm{X} = \frac{s_2(b+c)}{a}$$

$$\mathrm{LX} = \frac{2ss_1(b-c)}{a(b+c)} \qquad \mathrm{L'X} = \frac{2s_2s_3(b+c)}{a(b-c)}$$

(1) $$A'X \cdot A'L = A'D^2 = A'D_1^2 = \tfrac{1}{4}(b-c)^2$$

Because A C Q X are concyclic

therefore $\angle A'XQ = \tfrac{1}{2}A = \angle A'QL$;

therefore triangles A'XQ A'QL are similar ;

therefore $A'X : A'Q = A'Q : A'L$;

therefore
$$\begin{aligned} A'X \cdot A'L &= A'Q^2 \\ &= A'D^2 \end{aligned}$$

(2) $$A'X \cdot A'L' = A'D_2^2 = A'D_3^2 = \tfrac{1}{4}(b+c)^2$$

This follows, in a manner analogous to the preceding, from the similarity of triangles A'XQ' A'Q'L'.

The following method may be used for proving (1) and (2).

Since the points A I L I_1 form a harmonic range,
therefore their projections on BC will form a harmonic range :
that is, X D L D_1 is a harmonic range.
Hence, since DD_1 is bisected at A',

$$A'X \cdot A'L = A'D^2 = A'D_1^2.$$

Similarly, since I_2 A I_3 L' form a harmonic range, so also will D_2 X D_3 L'.

[redacted]

(5) $$A'X \cdot LD = A'D \cdot DX = \frac{s_1(b-c)^2}{2a}$$

For
$$\begin{aligned} A'X \cdot LD &= A'X \cdot A'D - A'X \cdot A'L \\ &= A'X \cdot A'D - A'D^2 \\ &= A'D \cdot DX \end{aligned}$$

(6) $$A'X \cdot LD_1 = A'D_1 \cdot D_1X = \frac{s(b-c)^2}{2a}$$

For
$$\begin{aligned} A'X \cdot LD_1 &= A'X \cdot A'D_1 + A'X \cdot A'L \\ &= A'X \cdot A'D_1 + A'D_1^2 \\ &= A'D_1 \cdot D_1X \end{aligned}$$

(7) $$A'X \cdot L'D_3 = A'D_2 \cdot D_2X = \frac{s_3(b+c)^2}{2a}$$

For
$$\begin{aligned} A'X \cdot L'D_2 &= A'X \cdot A'L' + A'X \cdot A'D_2 \\ &= \quad A'D_2^2 \quad + A'X \cdot A'D_2 \\ &= A'D_2 \cdot D_2X \end{aligned}$$

(8) $$A'X \cdot L'D_3 = A'D_3 \cdot D_3X = \frac{s_2(b+c)^2}{2a}$$

For
$$\begin{aligned} A'X \cdot L'D_3 &= A'X \cdot A'L' - A'X \cdot A'D_3 \\ &= \quad A'D_3^2 \quad - A'X \cdot A'D_3 \\ &= A'D_3 \cdot D_3X \end{aligned}$$

(9) $$A'L \quad LX = DL \cdot LD_1 = \frac{ss_1(b-c)^2}{(b+c)^2}$$

For
$$\begin{aligned} A'L \quad LX &= A'L \cdot A'X - A'L^2 \\ &= \quad A'D^2 \quad - A'L^2 \\ &= DL \cdot LD_1 \end{aligned}$$

(10) $$A'L' \cdot L'X = D_2L' \cdot L'D_3 = \frac{s_2s_3(b+c)^2}{(b-c)^2}$$

For
$$\begin{aligned} A'L' \cdot L'X &= A'L'^2 - A'L' \cdot A'X \\ &= A'L'^2 - A'D_3^2 \\ &= D_2L' \cdot L'D_3 \end{aligned}$$

(11) $$DX \cdot D_1X = BD \cdot DC - BX \cdot XC$$

For $$BD \cdot DC - BX \cdot XC = (A'B^2 - A'D^2) - (A'B^2 - A'X^2)$$
$$= A'X^2 - A'D^2$$
$$= DX \cdot D_1X$$

(12) $$D_2X \cdot D_3X = BD_2 \cdot D_2C + BX \cdot XC$$

For $$BD_2 \cdot D_2C - BX \cdot XC = (A'D_2^2 - A'C^2) + (A'C^2 - A'X^2)$$
$$= A'D_2^2 - A'X^2$$
$$= D_2X \cdot D_3X$$

In ***Mathematical Questions from the Educational Times***, XIII. 34 (1870), T. T. Wilkinson says regarding (1):

"This is one of the properties of Halley's diagram, which was partially discussed in the four numbers of the *Student*, published at Liverpool from 1797 to 1800. It there forms Prop. 8, and is due to *Non Sibi*, a name assumed by the first editor, Mr John Knowles. In the diagram as there considered, the properties of one side only are given; but when all the sides are considered, there seems to be no limit to the relations between the different parts of the figure. Some time ago I considered the 'angular properties' only; and after writing down about 130 of [redacted]

§ 4.

To find values for the rectangles contained by various segments of the diameter UU′.

FIGURE 15.

(1) $$A'U \cdot UK' = A'X \cdot A'L = \tfrac{1}{4}(b - c)$$

From the similar triangles UA′L AKU′

$$A'L : A'U = KU' : KA$$

that is, $$A'L : A'U = UK' : A'X$$

(2) $$A'U' \cdot U'K' = A'X \cdot A'L' = \tfrac{1}{4}(b + c)^2$$

From the similar triangles UA′L′ AKU

$$A'L' : A'U' = KU : KA$$

that is, $$A'L' : A'U' = U'K' : A'X$$

(3) $$A'K \cdot KU' = A'X \cdot LX$$

For
$$\begin{aligned} A'X \cdot LX &= DX \cdot D_1X \\ &= A'X^2 - A'D^2 \\ &= KA^2 - A'D^2 \\ &= UK \cdot KU' - A'U \cdot KU' \\ &= A'K \cdot KU' \end{aligned}$$

(4) $$A'K \cdot KU = A'X \cdot L'X$$

For
$$\begin{aligned} A'X \cdot L'X &= D_2X \cdot D_3X \\ &= A'D_2^2 - A'X^2 \\ &= A'D_2^2 - KA^2 \\ &= A'U' \cdot U'K' - UK \cdot KU' \\ &= A'K \cdot KU \end{aligned}$$

(5) $$A'K \cdot A'U = BD \cdot DC$$

For $$\begin{aligned} BD \cdot DC &= A'C^2 - A'D^2 \\ &= A'U \cdot A'U' - A'U \cdot UK' \\ &= A'K \cdot A'U \end{aligned}$$

(6) $$A'K \cdot A'U' = BD_2 \cdot D_2C$$

For $$\begin{aligned} BD_2 \cdot D_2C &= A'D_2^2 - A'C^2 \\ &= A'U' \cdot U'K' - A'U \cdot A'U' \\ &= A'K \cdot A'U' \end{aligned}$$

(7) $$A'K \cdot A'K' = BX \cdot XC$$

For $$\begin{aligned} BX \cdot XC &= AX \cdot XR \\ &= A'K \cdot A'K' \end{aligned}$$

(8)* $$A'U \cdot UK = US^2$$

For $$\begin{aligned} US^2 &= CU^2 - CS^2 \\ &= A'U \cdot UU' - A'U \cdot UK' \\ &= A'U \cdot U'K' \\ &= A'U \cdot UK \end{aligned}$$

(9) $$U'K \cdot ID = A'D \cdot DX$$

[redacted]

(11) $$ABC : XPQ = UU' : U'K$$
and $$ABC : XP'Q' = UU' : UK$$

FIGURE 9.

Draw $A'X'$ perpendicular to PQ;
then X' is the mid point of PQ.

Because $\angle UCA' = \frac{1}{2}A = \angle A'PX'$,
therefore the right-angled triangles $UA'C$ $A'X'P$ are similar;

therefore $$A'C^2 : X'P^2 = UC^2 : A'P^2.$$

But $$\begin{aligned} ABC : XPQ &= BC^2 : PQ^2 \\ &= A'C^2 : X'P^2; \end{aligned}$$

therefore $$\begin{aligned} ABC : XPQ &= UC^2 : A'P^2 \\ &= A'U \cdot UU' : A'U \cdot UK' \\ &= UU' : UK' \\ &= UU' : U'K \end{aligned}$$

For another proof see § 1, (28).

In a similar manner it may be shown that

$$ABC : XP'Q' = UU' : UK.$$

(12) Because $U'K + UK = UU'$
another proof is obtained of the theorem that

$$ABC = XPQ + XP'Q'$$

(13) If the base BC and the vertical angle A be given, and if in AU AU' the bisectors of the interior and exterior angles at A, there be taken AP equal to half the sum, and AQ equal to half the difference of the sides, the loci of P and Q are two circles. If their radii be denoted by r' r'' and the radius of the circle inscribed in CUU' by r''', then

$$R = r' + r'' + r'''$$

Mr G. Robinson, jun., Hexham, in the *Lady's and Gentleman's Diary* for 1862, p. 74. Two solutions will be found in the *Diary* for 1863, pp. 49-50.

§ 5.

If through A' a perpendicular is drawn to BC, then AD AD_1 AD_2 AD_3 will intersect this perpendicular at R_1 R R_3 R_2 such that*

$$A'R = r \quad A'R_1 = r_1 \quad A'R_2 = r_2 \quad A'R_3 = r_3$$

Figure 16.

Let DI produced meet AD_1 at D'.

Since the line joining the extremities of two parallel and similarly directed radii of two circles passes through their external homothetic centre; and since A is the external homothetic centre of the circles I_1 and I, and I_1D_1 ID' are parallel; therefore ID' is a radius of the incircle I, and $DD' = 2r$.

Now since $A'D = A'D_1$, and $A'R$ is parallel to DD', therefore
$$A'R = \tfrac{1}{2}DD' = r.$$

Similarly for the other equalities

(1) Through B', the mid point of CA, a perpendicular to

[redacted]

(2) The four triangles RST $R_1S_1T_1$ $R_2S_2T_2$ $R_3S_3T_3$ are inversely similar to ABC; they have O, the circumcentre of ABC, for their common centre of homology, and OI OI_1 OI_2 OI_3 for the diameters of their circumcircles.

FIGURE 17.

Since $$A'R_1 = r_1 = I_1D_1$$
therefore $\angle I_1R_1A'$ is right.
Similarly $\angle I_1S_1B'$ and $\angle I_1T_1C'$ are right;
therefore the circle whose diameter is OI_1
passes through R_1 S_1 T_1.

Since R_1 S_1 O_1 T_1 are concyclic,

therefore
$$\begin{aligned} \angle S_1R_1T_1 &= 180^\circ - \angle S_1OT_1 \\ &= 180^\circ - \angle B'OC' \\ &= A; \end{aligned}$$

and
$$\angle R_1S_1T_1 = \angle R_1OT_1 = B,$$

since T_1O R_1O are respectively perpendicular to AB BC;
therefore triangle $R_1S_1T_1$ is similar to ABC.

(3) Let the mid points of OI OI_1 OI_2 OI_3
be denoted by I' I_1' I_2' I_3'
then $I_1'I_2'I_3'I'$ is an orthic tetrastigm, similar and similarly situated to the tetrastigm $I_1I_2I_3I$, and the radius of the circumcircle of any of its four triangles is R.

For the radius of the circumcircle of any of the four triangles of the orthic tetrastigm $I_1I_2I_3I$ is $2R$.

(4) Let AI BI CI meet the circumcircle of ABC in U V W, and let the points diametrically opposite to U V W be U' V' W'.

Then I is the orthocentre of the triangle UVW. Now since O is the circumcentre of UVW, therefore I' is the nine-point centre of the four triangles of the orthic tetrastigm $UVWI$.

In like manner since I_1 is the orthocentre, and O the circumcentre of the triangle $UV'W'$, I_1' is the nine-point centre of the four triangles of the orthic tetrastigm $UV'W'I_1$; and similarly for I_2' I_3'.

See *Proceedings of the Edinburgh Mathematical Society*, Vol. I., pp. 54–5 (1894).

(5) The sum of the circumcircles of the four RST triangles is three times the circumcircle of ABC.

Since circles are proportional to the squares of their diameters, the circumcircle of ABC is to the sum of the four RST circles as $4R^2$ is to $OI^2 + OI_1^2 + OI_2^2 + OI_3^2$.

Now
$$\begin{aligned} \Sigma(OI^2) &= 4R^2 + 2R(r_1 + r_2 + r_3 - r) \\ &= 4R^2 + 2R \cdot 4R \\ &= 12R^2. \end{aligned}$$

§ 6.

If UD UD_1 $U'D_2$ $U'D_3$ intersect AX at X_0 X_1 X_2 X_3, then*
$$XX_0 = r \quad XX_1 = r_1 \quad XX_2 = r_2 \quad XX_3 = r_3$$

[redacted]

Similarly, if through I_1 a parallel be drawn to BC meeting AX in X_1, it may be proved that $U\ D_1\ X_1$ are collinear.

Through I_3 draw a parallel to BC meeting UU' in K_3 and AX in X_3; join $U'D_3\ D_3X_3$.

Because $$A'D_3^2 = A'X \cdot A'L'$$

therefore $$A'D_3 : A'X = A'L' : A'D_3$$

that is $$\begin{aligned} A'D_3 : K_3X_3 &= A'L' : K_3I_3 \\ &= U'A' : U'K_3\,; \end{aligned}$$

therefore the points $U'\ D_3\ X_3$ are collinear.

Similarly for the points $U'\ D_2\ X_2$.

(1) $VE\ \ V'E_1\ \ VE_2\ \ V'E_3$ intersect BY at $Y_0\ \ Y_1\ \ Y_2\ \ Y_3$; and

$WF\ \ W'F_1\ \ W'E_2\ \ WF_3$ intersect CZ at $Z_0\ \ Z_1\ \ Z_2\ \ Z_3$ such that

$$YY_0 = r \quad YY_1 = r_1 \quad YY_2 = r_2 \quad YY_3 = r_3$$
$$ZZ_0 = r \quad ZZ_1 = r_1 \quad ZZ_2 = r_2 \quad ZZ_3 = r_3.$$

(2) The four triangles $X_0Y_0Z_0\ \ X_1Y_1Z_1\ \ X_2Y_2Z_2\ \ X_3Y_3Z_3$ are inversely similar to ABC; they have H, the orthocentre of ABC, for their common centre of homology, and $HI\ HI_1\ HI_2\ HI_3$ for the diameters of their circumcircles.

FIGURE 19.

Since $$XX_1 = r_1 = I_1D_1$$
therefore $\angle\, I_1X_1X$ is right.
Similarly $\angle\, I_1Y_1Y$ and $\angle\, I_1Z_1Z$ are right;
therefore the circle whose diameter is HI
passes through $X_1\ Y_1\ Z_1$.

Since X_1 Y_1 H Z_1 are concyclic

therefore $\angle Y_1X_1Z_1 = 180° - \angle Y_1HZ_1$

$= A$;

and $\angle X_1Y_1Z_1 = \angle X_1HZ_1 = B$,

since Z_1H X_1H are respectively perpendicular to AB BC ;

therefore triangle $X_1Y_1Z_1$ is similar to ABC.

(3) The mid points of HI HI_1 HI_2 HI_3 form an orthic tetrastigm similar and similarly situated to the tetrastigm $I_1I_2I_3I$, and the radius of the circumcircle of any of its four triangles is R.

(4) The sum of the circumcircles of the four triangles $X_0Y_0Z_0$ .... is four times the sum of the circumcircles of the three triangles

AYZ XBZ XYC.

It will be seen from a subsequent Section that the values * of HI^2 .... may be written

$$HI^2 = 4(R^2 - 2Rr) + bc + ca + ab - (a^2 + b^2 + c^2)$$
$$HI_1^2 = 4(R^2 + 2Rr_1) + bc - ca - ab - (a^2 + b^2 + c^2)$$
$$HI_2^2 = 4(R^2 + 2Rr_2) - bc + ca - ab - (a^2 + b^2 + c^2)$$
$$HI_3^2 = 4(R^2 + 2Rr_3) - bc - ca + ab - (a^2 + b^2 + c^2)$$

[redacted]

The triangles CBZ AHZ are similar;

therefore $BC^2 : HA^2 = CZ^2 : AZ^2$;

therefore $BC^2 + HA^2 : BC^2 = CZ^2 + AZ^2 : CZ^2$

$= CA^2 : CZ^2$;

therefore $BC^2 + HA^2 = \dfrac{BC^2 \cdot CA^2}{CZ^2}$

$= 4R^2$

by a theorem of Brahmegupta.

Similarly $CA^2 + HB^2 = AB^2 + HC^2 = 4R^2$

For another proof see Feuerbach, *Eigenschaften...des...Dreiecks*, Section VI., Theorem 2.

§ 7.

If $A'I$ $A'I_1$ $A'I_2$ $A'I_3$ intersect AX at

X_0 X_1 X_2 X_3 then*

$$AX_0 = r \quad AX_1 = r_1 \quad AX_2 = r_2 \quad AX_3 = r_3$$

FIGURE 21.

Join CU, and draw the radius of the incircle IE.

Then $\angle UCA' = \angle IAE$;

therefore triangles CUA′ AIE are similar;

therefore $CU : UA' = AI : IE$.

Now $CU : UA' = IU : UA'$

$= AI : AX_0$;

therefore $AX_0 = IE = r$

Similarly $AX_1 = r_1$.

* The first of these properties occurs incidentally in William Walker's proof of a theorem in the *Gentleman's Mathematical Companion* for 1803, p. 50.

If CU' be joined, and I_3E_3 the radius of the third excircle be drawn, then triangles $CU'A'$ AI_3E_3 will be similar, and since

$$CU' = I_3U',$$

it may be shown that $\quad AX_3 = I_3E_3 = r_3.$

Corresponding to the four X points situated on AX, there will be four Y points, $Y_0\ Y_1\ Y_2\ Y_3$, situated on BY, and four Z points, $Z_0\ Z_1\ Z_2\ Z_3$ situated on CZ.

Some of the properties of this collection of points will be found in the *Proceedings of the Edinburgh Mathematical Society*, Vol. I., pp. 89–96 (1894).

§ 8.

Figure 23.

If the medians AA' BB' CC' be intersected by

the radii			at the points		
DI	EI	FI	L_0	M_0	N_0
D_1I_1	E_1I_1	F_1I_1	L_1	M_1	N_1
D_2I_2	E_2I_2	F_2I_2	L_2	M_2	N_2
D_3I_3	E_3I_3	F_3I_3	L_3	M_3	N_3

then*

[redacted]

it is on the opposite side from A. A similar convention holds for the M and the N points.

FIGURE 22.

Let AI meet BC at L; draw LS LT perpendicular to AC AB, and AX perpendicular to BC.

Then $$A'L : A'D = A'D : A'X$$
$$= L_0D : AX.$$

Now $$A'L : A'D = A'D - A'L : A'X - A'D$$
$$= LD : DX$$
$$= LI : IA$$
$$= LB : BA$$
$$= LT : AX;$$

therefore $$L_0D = LT = LS$$

Again $$LS \cdot AC + LT \cdot AB = 2ABC$$

therefore $$L_0D(b + c) = 2\triangle$$

Similarly for the other equalities

(1) L_0 M_0 N_0 lie on EF FD DE; and similarly for L_1 M_1 N_1 .....

A proof of this will be found in the *Proceedings of the Edinburgh Mathematical Society*, Vol. I., pp. 57–8 (1894).

(2) $$\frac{1}{DL_0} + \frac{1}{D_1L_1} + \frac{1}{D_2L_2} + \frac{1}{D_3L_3}$$
$$= \frac{1}{EM_0} + \frac{1}{E_1M_1} + \frac{1}{E_2M_2} + \frac{1}{E_3M_3}$$
$$= \frac{1}{FN_0} + \frac{1}{F_1N_1} + \frac{1}{F_2N_2} + \frac{1}{F_3N_3} = 0$$

(3)*

$$\frac{1}{DL_0}+\frac{1}{EM_0}+\frac{1}{FN_0}=\frac{2}{r}$$

$$\frac{1}{D_1L_1}+\frac{1}{E_1M_1}+\frac{1}{F_1N_1}=-\frac{2}{h_1}$$

$$\frac{1}{D_2L_2}+\frac{1}{E_2M_2}+\frac{1}{F_2N_2}=-\frac{2}{h_2}$$

$$\frac{1}{D_3L_3}+\frac{1}{E_3M_3}+\frac{1}{F_3N_3}=-\frac{2}{h_3}$$

(4) The diagonals of the following pairs of parallelograms

$DL_0D_1L_1$	$D_2L_2D_3L_3$	intersect	at	A′
$EM_0E_2M_2$	$E_3M_3E_1M_1$	,,	,,	B′
$FN_0F_3N_3$	$F_1N_1F_2N_2$	,,	,,	C′

(5) The four LMN triangles are homologous, and their centre of homology is G the centroid of ABC.

(6)† $A'U : A'U - IL_0 = A'U : I_1L_1 - A'U = b+c : a$

$A'U' : I_2L_2 + A'U' = A'U' : I_3L_3 - A'U' = b-c : a$

FIGURE 23.

From I U draw IE US perpendicular to AC.

[redacted]

(8) If through $I\ I_1\ I_2\ I_3$ parallels be drawn to BC, meeting UU' in $K\ K_1\ K_2\ K_3$, then*

$$UK = UK_1 = US$$
$$U'K_2 = U'K_3 = U'S'$$

where $US\ U'S'$ are perpendicular to AC.

For the right-angled triangles $CUS\ IUK$ are congruent,

since $$UC = UI,$$

and $$\angle CUS = \tfrac{1}{2}(B - C) = \angle IUK.$$

§ 9.

Formulae connected with the Angular Bisectors of a Triangle limited at their points of intersection with each other.

The notation

$$AI = \alpha \quad AI = \beta \quad CI = \gamma, \quad \text{etc.},$$

was suggested by T. S. Davies in the *Lady's and Gentleman's Diary* for 1842, p. 77, and adopted by Thomas Weddle in his admirable papers entitled "Symmetrical Properties of Plane Triangles," which appeared in the same publication (1843, 1845, 1848).

Neither Davies nor Weddle makes use of the equivalents for II_1, etc., namely $\alpha_1 - \alpha$, etc. Although the employment of these equivalents somewhat lengthens the formulae, it seems to me that it renders their symmetry a little more apparent.

In connection with the ascription, in the historical notes, of the great majority of the following formulae to Weddle, it is right to call attention to a letter of T. S. Davies in the *Lady's and Gentleman's Diary* for 1849, pp. 90-1, in which he states that when he undertook to arrange and systematise those properties of the triangle communicated to him, several sets of papers came into his hands, the most ample and elegant of which were those of Messrs Weddle and J. W. Elliott. The letter continues:

"I feel it to be due to him [Mr Elliott] to say that the names both of Mr Weddle and Mr Elliott might fairly have been prefixed to the far greater number of the properties, whilst each gentleman would have had a few properties designated as peculiar to himself."

I might have considerably shortened the lists of the formulae by giving only the leading identities, and referring the reader to Mr Lemoine's scheme of *continuous transformation*. I have done so here and there, but in general I have

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* The property that $UK = US$ is referred to as well known in the *Gentleman's Mathematical Companion* for 1803, p. 50.

either left or made the list complete in order that the reader who consults it may find ready to his hand the particular scrap of information of which he is in search. A table of logarithms which gave the logarithms of the prime numbers only would certainly be of some use, but merely to a select few.

Mr R. F. Muirhead suggests to me $t_1\ t_2\ t_3$ as a convenient mathematical translation of Mr Lemoine's *transformation continue en A, en B, en C*. Here is the t_1 applicable to the formulae which follow.

$$a \quad b \quad c \quad s \quad s_1 \quad s_2 \quad s_3 \quad r \quad r_1 \quad r_2 \quad r_3 \quad h_1 \quad h_2 \quad h_3$$

change into

$$a \quad -b \quad -c \quad -s_1 \quad -s \quad s_3 \quad s_2 \quad r_1 \quad r \quad -r_3 \quad -r_2 \quad -h_1 \quad h_3 \quad h_2$$

$$\mathrm{A} \quad \mathrm{B} \quad \mathrm{C} \quad \mathrm{R} \quad \triangle \quad l_1 \quad l_2 \quad l_3 \quad \lambda_1 \quad \lambda_2 \quad \lambda_3$$

change into

$$-\mathrm{A} \quad 180^\circ - \mathrm{B} \quad 180^\circ - \mathrm{C} \quad -\mathrm{R} \quad -\triangle \quad -l_1 \quad -\lambda_3 \quad -\lambda_2 \quad \lambda_1 \quad -l_2 \quad -l$$

$$\alpha \quad \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \beta \quad \beta_1 \quad \beta_2 \quad \beta_3 \quad \gamma \quad \gamma_1 \quad \gamma_2 \quad \gamma_3$$

change into

$$-\alpha_1 \quad -\alpha \quad -\alpha_3 \quad -\alpha_2 \quad \beta_1 \quad \beta \quad -\beta_3 \quad -\beta_2 \quad \gamma_1 \quad \gamma \quad -\gamma_3 \quad -\gamma_2$$

$$a^2 = (r_2 - r)(r_3 - r) \qquad a_1^2 = (r_1 + r_1)(r_1 + r_2)$$

$$\left.\begin{aligned}
\alpha &= \frac{\sqrt{bcrr_1}}{r_1} & \beta &= \frac{\sqrt{carr_2}}{r_2} & \gamma &= \frac{\sqrt{abrr_3}}{r_3} \\
\alpha_1 &= \frac{\sqrt{bcrr_1}}{r} & \beta_2 &= \frac{\sqrt{carr_2}}{r} & \gamma_3 &= \frac{\sqrt{abrr_3}}{r} \\
\alpha_2 &= \frac{\sqrt{bcr_2r_3}}{r_3} & \beta_3 &= \frac{\sqrt{car_3r_1}}{r_1} & \gamma_1 &= \frac{\sqrt{abr_1r_2}}{r_2} \\
\alpha_3 &= \frac{\sqrt{bcr_2r_3}}{r_2} & \beta_1 &= \frac{\sqrt{car_3r_1}}{r_3} & \gamma_2 &= \frac{\sqrt{abr_1r_2}}{r_1}
\end{aligned}\right\} \tag{2}$$

$$\left.\begin{aligned}
&\frac{\alpha}{r} = \frac{\alpha_1}{r_1} = \frac{\alpha_2}{s_3} = \frac{\alpha_3}{s_2} && \frac{\alpha}{s_1} = \frac{\alpha_1}{s} = \frac{\alpha_2}{r_2} = \frac{\alpha_3}{r_3} \\
&\frac{\beta}{r} = \frac{\beta_1}{s_3} = \frac{\beta_2}{r_2} = \frac{\beta_3}{s_1} && \frac{\beta}{s_2} = \frac{\beta_1}{r_1} = \frac{\beta_2}{s} = \frac{\beta_3}{r_3} \\
&\frac{\gamma}{r} = \frac{\gamma_1}{s_2} = \frac{\gamma_2}{s_1} = \frac{\gamma_3}{r_3} && \frac{\gamma}{s_3} = \frac{\gamma_1}{r_1} = \frac{\gamma_2}{r_2} = \frac{\gamma_3}{s}
\end{aligned}\right\} \tag{3}$$

$$\alpha\alpha_1 = \alpha_2\alpha_3 = bc \quad \beta\beta_2 = \beta_3\beta_1 = ca \quad \gamma\gamma_3 = \gamma_1\gamma_2 = ab \tag{4}$$

$$\alpha\alpha_1\alpha_2\alpha_3 = b^2c^2 \quad \beta\beta_1\beta_2\beta_3 = c^2a^2 \quad \gamma\gamma_1\gamma_2\gamma_3 = a^2b^2 \tag{5}$$

$$\alpha\beta\gamma\alpha_1\beta_1\gamma_1\alpha_2\beta_2\gamma_2\alpha_3\beta_3\gamma_3 = (abc)^4 \tag{6}$$

$$\left.\begin{aligned}
&\alpha\,\beta_1\gamma_3 = \alpha\,\beta_2\gamma_1 = \alpha_1\beta\,\gamma_2 = \alpha_1\beta_3\gamma \\
&= \alpha_2\beta\,\gamma_3 = \alpha_2\beta_3\gamma_1 = \alpha_3\beta_1\gamma_2 = \alpha_3\beta_2\gamma = abc
\end{aligned}\right\} \tag{7}$$

$$\left.\begin{aligned}
\alpha\,\beta\,\gamma &: abc = abc : \alpha_1\beta_2\gamma_3 \\
\alpha_1\beta_1\gamma_1 &: abc = abc : \alpha\,\beta_3\gamma_2 \\
\alpha_2\beta_2\gamma_2 &: abc = abc : \alpha_3\beta\,\gamma_1 \\
\alpha_3\beta_3\gamma_3 &: abc = abc : \alpha_2\beta_1\gamma
\end{aligned}\right\} \tag{8}$$

$$\left.\begin{aligned}
&\frac{\beta\,\gamma_2}{\alpha}=\frac{\beta_3\gamma}{\alpha}=\frac{\beta_1\gamma_3}{\alpha_1}=\frac{\beta_2\gamma_1}{\alpha_1}\\
&=\frac{\beta_1\gamma_2}{\alpha_2}=\frac{\beta_2\gamma}{\alpha_2}=\frac{\beta\,\gamma_3}{\alpha_3}=\frac{\beta_3\gamma_1}{\alpha_3}=a\\
&\frac{\alpha\,\gamma_1}{\beta}=\frac{\alpha_3\gamma}{\beta}=\frac{\alpha_1\gamma}{\beta_1}=\frac{\alpha_2\gamma_1}{\beta_1}\\
&=\frac{\alpha_1\gamma_3}{\beta_2}=\frac{\alpha_2\gamma_3}{\beta_2}=\frac{\alpha\,\gamma_2}{\beta_3}=\frac{\alpha_3\gamma_2}{\beta_3}=b\\
&\frac{\alpha\,\beta_1}{\gamma}=\frac{\alpha_2\beta}{\gamma}=\frac{\alpha_1\beta}{\gamma_1}=\frac{\alpha_3\beta_1}{\gamma_1}\\
&=\frac{\alpha\,\beta_2}{\gamma_2}=\frac{\alpha_2\beta_3}{\gamma_2}=\frac{\alpha_1\beta_3}{\gamma_3}=\frac{\alpha_3\beta_2}{\gamma_3}=c
\end{aligned}\right\}\qquad(9)$$

$$\left.\begin{aligned}
&\alpha : \beta = \gamma : r_1 - r && \alpha_1 : \beta_1 = \gamma_1 : r_1 - r\\
&\beta : \alpha = \gamma : r_2 - r && \beta_1 : \alpha_1 = \gamma_1 : r_2 + r_1\\
&\gamma : \alpha = \beta : r_3 - r && \gamma_1 : \alpha_1 = \beta_1 : r_1 + r_2\\
&\alpha_2 : \beta_2 = \gamma_2 : r_2 + r_3 && \alpha_3 : \beta_3 = \gamma_3 : r_2 + r_3\\
&\beta_2 : \alpha_2 = \gamma_2 : r_2 - r && \beta_3 : \alpha_3 = \gamma_3 : r_3 + r_1\\
&\gamma_2 : \alpha_2 = \beta_2 : r_1 + r_2 && \gamma_3 : \alpha_3 = \beta_3 : r_3 - r
\end{aligned}\right\}\qquad(10)$$

$$\alpha_1 : \beta_2 = \gamma_2 : r_2 + r_3 \qquad \alpha : \beta_3 = \gamma_3 : r_2 + r_3$$

$$\left.\begin{aligned}
\frac{\alpha^2}{bc} + \frac{\beta^2}{ca} + \frac{\gamma^2}{ab} &= 1\\
\frac{\alpha_1^2}{bc} - \frac{\beta_1^2}{ca} - \frac{\gamma_1^2}{ab} &= 1\\
-\frac{\alpha_2^2}{bc} + \frac{\beta_2^2}{ca} - \frac{\gamma_2^2}{ab} &= 1\\
-\frac{\alpha_3^2}{bc} - \frac{\beta_3^2}{ca} + \frac{\gamma_3^2}{ab} &= 1
\end{aligned}\right\} \qquad (13)$$

$$\left.\begin{aligned}
\frac{bc}{\alpha_1^2} + \frac{ca}{\beta_2^2} + \frac{ab}{\gamma_3^2} &= 1\\
\frac{bc}{\alpha^2} - \frac{ca}{\beta_3^2} - \frac{ab}{\gamma_2^2} &= 1\\
-\frac{bc}{\alpha_3^2} + \frac{ca}{\beta^2} - \frac{ab}{\gamma_1^2} &= 1\\
-\frac{bc}{\alpha_2^2} - \frac{ca}{\beta_1^2} + \frac{ab}{\gamma^2} &= 1
\end{aligned}\right\} \qquad (14)$$

$$\left.\begin{aligned}
\alpha^2\left(\frac{1}{c} - \frac{1}{b}\right) + \beta^2\left(\frac{1}{a} - \frac{1}{c}\right) + \gamma^2\left(\frac{1}{b} - \frac{1}{a}\right) &= 0\\
\alpha_1^2\left(\frac{1}{b} - \frac{1}{c}\right) + \beta_1^2\left(\frac{1}{a} + \frac{1}{c}\right) - \gamma_1^2\left(\frac{1}{a} + \frac{1}{b}\right) &= 0
\end{aligned}\right\} \qquad (15)$$

$$\left.\begin{aligned}
a\alpha^2(b-c) + b\beta^2(c-a) + c\gamma^2(a-b) &= 0\\
a\alpha_1^2(c-b) + b\beta_1^2(c+a) + c\gamma_1^2(a+b) &= 0
\end{aligned}\right\} \qquad (16)$$

$$\left.\begin{aligned}
\frac{b-c}{a\alpha_1^2} + \frac{c-a}{b\beta_2^2} + \frac{a-b}{c\gamma_3^2} &= 0\\
\frac{c-b}{a\alpha^2} + \frac{c+a}{b\beta_3^2} - \frac{a+b}{c\gamma_2^2} &= 0
\end{aligned}\right\} \qquad (17)$$

$$\left.\begin{aligned}
\alpha^2+\beta^2+\gamma^2 &= \frac{bcs_1+cas_2+abs_3}{s} = bc+ca+ab-\frac{3abc}{s}\\
\alpha_1^2+\beta_1^2+\gamma_1^2 &= \frac{bcs+cas_3+abs_2}{s_1} = bc-ca-ab+\frac{3abc}{s_1}\\
\alpha_2^2+\beta_2^2+\gamma_2^2 &= \frac{bcs_3+cas+abs_1}{s_2} = bc+ca-ab+\frac{3abc}{s_2}\\
\alpha_3^2+\beta_3^2+\gamma_3^2 &= \frac{bcs_2+cas_1+abs}{s_3} = -bc-ca+ab+\frac{3abc}{s_3}
\end{aligned}\right\} \quad (18)$$

$$\left.\begin{aligned}
\alpha_1^2+\beta_2^2+\gamma_3^2 &= (r_1+r_2+r_3)^2+s^2\\
\alpha^2+\beta_3^2+\gamma_2^2 &= (r-r_3-r_2)^2+s_1^2\\
\alpha_3^2+\beta^2+\gamma_1^2 &= (r-r_1-r_3)^2+s_2^2\\
\alpha_2^2+\beta_1^2+\gamma^2 &= (r-r_2-r_1)^2+s_3^2
\end{aligned}\right\} \quad (19)$$

Compare (15) of the r formulae*

$$\left.\begin{aligned}
\alpha\beta\gamma : abc &= r : s\\
\alpha_1\beta_1\gamma_1 : abc &= r_1 : s_1\\
\alpha_2\beta_2\gamma_2 : abc &= r_2 : s_2\\
\alpha_3\beta_3\gamma_3 : abc &= r_3 : s_3
\end{aligned}\right\} \quad (20)$$

Other proportions may be obtained by substituting for abc its equivalents in (7). Matthes (p. 49) gives

$$\left.\begin{aligned}
a\alpha^2 + b\beta^2 + c\gamma^2 &= \\
a\alpha_1^2 - b\beta_1^2 - c\gamma_1^2 &= \\
-a\alpha_2^2 + b\beta_2^2 - c\gamma_2^2 &= \\
-a\alpha_3^2 - b\beta_3^2 + c\gamma_3^2 &= abc
\end{aligned}\right\} \quad (23)$$

$$\left.\begin{aligned}
\alpha\,\beta\,\gamma &= (r_1 - r)(r_2 - r)(r_3 - r) \\
\alpha_1\beta_1\gamma_1 &= (r_1 - r)(r_3 + r_1)(r_1 + r_2) \\
\alpha_2\beta_2\gamma_2 &= (r_2 + r_3)(r_2 - r)(r_1 + r_2) \\
\alpha_3\beta_3\gamma_3 &= (r_2 + r_3)(r_3 + r_1)(r_3 - r) \\[6pt]
\alpha_1\beta_2\gamma_3 &= (r_2 + r_3)(r_3 + r_1)(r_1 + r_2) \\
\alpha\,\beta_3\gamma_2 &= (r_2 + r_3)(r_2 - r)(r_3 - r) \\
\alpha_3\beta\,\gamma_1 &= (r_1 - r)(r_3 + r_1)(r_3 - r) \\
\alpha_2\beta_1\gamma &= (r_1 - r)(r_2 - r)(r_1 + r_2)
\end{aligned}\right\} \quad (24)$$

$$\left.\begin{aligned}
a\,\beta\,\beta_3 &= (r_2 + r_3)(r_3 - r)(r_1 - r) \\
a\,\gamma\,\gamma_2 &= (r_2 + r_3)(r_1 - r)(r_2 - r) \\
b\,\gamma\,\gamma_1 &= (r_3 + r_1)(r_1 - r)(r_2 - r) \\
b\,\alpha\,\alpha_3 &= (r_3 + r_1)(r_2 - r)(r_3 - r) \\
c\,\alpha\,\alpha_2 &= (r_1 + r_2)(r_2 - r)(r_3 - r) \\
c\,\beta\,\beta_1 &= (r_1 + r_2)(r_3 - r)(r_1 - r) \\[6pt]
a\,\beta_1\beta_2 &= (r_1 - r)(r_1 + r_2)(r_2 + r_3) \\
a\,\gamma_3\gamma_1 &= (r_1 - r)(r_2 + r_3)(r_3 + r_1) \\
b\,\gamma_2\gamma_3 &= (r_2 - r)(r_2 + r_3)(r_3 + r_1) \\
b\,\alpha_1\,\alpha_2 &= (r_2 - r)(r_3 + r_1)(r_1 + r_2) \\
c\,\alpha_3\,\alpha_1 &= (r_3 - r)(r_3 + r_1)(r_1 + r_2) \\
c\,\beta_2\beta_3 &= (r_3 - r)(r_1 + r_2)(r_2 + r_3)
\end{aligned}\right\} \quad (25)$$

Weddle remarks that (24) and (25) exhibit the twenty products of every three of the six quantities

$$r_1 - r, \quad r_2 - r, \quad r_3 - r, \quad r_2 + r_3, \quad r_3 + r_1, \quad r_1 + r_2$$

$$\left.\begin{array}{ll}
\alpha\,\alpha_2 = (r_2 - r)c & \alpha_3\,\alpha_1 = (r_3 + r_1)c \\
\alpha\,\alpha_3 = (r_3 - r)b & \alpha_1\,\alpha_2 = (r_1 + r_2)b \\
\beta\beta_3 = (r_3 - r)a & \beta_1\beta_2 = (r_1 + r_2)a \\
\beta\beta_1 = (r_1 - r)c & \beta_2\beta_3 = (r_2 + r_3)c \\
\gamma\gamma_1 = (r_1 - r)b & \gamma_2\gamma_3 = (r_2 + r_3)b \\
\gamma\gamma_2 = (r_2 - r)a & \gamma_3\gamma_1 = (r_3 + r_1)a
\end{array}\right\} \quad (26)$$

$$\left.\begin{array}{ll}
\alpha\,b = \alpha_3(r_3 - r) & \alpha_2 b = \alpha_1(r_3 - r) \\
\alpha\,c = \alpha_2(r_2 - r) & \alpha_2 c = \alpha\,(r_1 + r_2) \\
\beta\,c = \beta_1(r_3 - r) & \beta_2 c = \beta_3(r_1 + r_2) \\
\beta a = \beta_3(r_1 - r) & \beta_2 a = \beta_1(r_2 + r_3) \\
\gamma\,a = \gamma_2(r_1 - r) & \gamma_2 a = \gamma\,(r_2 + r_3) \\
\gamma\,b = \gamma_1(r_2 - r) & \gamma_2\,b = \gamma_3(r_2 - r) \\
\alpha_1 b = \alpha_2\,(r_3 + r_1) & \alpha_3\,b = \alpha\,(r_3 + r_1) \\
\alpha_1 c = \alpha_3\,(r_1 + r_2) & \alpha_3\,c = \alpha_1\,(r_2 - r) \\
\beta_1\,c = \beta\,(r_1 + r_2) & \beta_3\,c = \beta_2(r_3 - r) \\
\beta_1 a = \beta_2(r_1 - r) & \beta_3 a = \beta\,(r_2 + r_3) \\
\gamma_1 a = \gamma_3(r_1 - r) & \gamma_3 a = \gamma_1(r_2 + r_3) \\
\gamma_1\,b = \gamma\,(r_3 + r_1) & \gamma_3 b = \gamma_2(r_3 + r_1)
\end{array}\right\} \quad (27)$$

[redacted]

$$\left.\begin{aligned}
(\alpha_1 - \alpha)s_2 &= \alpha_3(r_1 - r) = \beta\gamma_1\\
(\alpha_1 - \alpha)s_3 &= \alpha_2(r_1 - r) = \beta_1\gamma\\
(\beta_2 - \beta)s_3 &= \beta_1(r_2 - r) = \gamma\alpha_2\\
(\beta_2 - \beta)s_1 &= \beta_3(r_2 - r) = \gamma_2\alpha\\
(\gamma_3 - \gamma)s_1 &= \gamma_2(r_3 - r) = \alpha\beta_3\\
(\gamma_3 - \gamma)s_2 &= \gamma_1(r_3 - r) = \alpha_3\beta\\
(\alpha_2 + \alpha_3)s &= \alpha_1(r_2 + r_3) = \beta_2\gamma_3\\
(\alpha_2 + \alpha_3)s_1 &= \alpha(r_2 + r_3) = \beta_3\gamma_2\\
(\beta_3 + \beta_1)s &= \beta_2(r_3 + r_1) = \gamma_3\alpha_1\\
(\beta_3 + \beta_1)s_2 &= \beta(r_3 + r_1) = \gamma_1\alpha_3\\
(\gamma_1 + \gamma_2)s &= \gamma_3(r_1 + r_2) = \alpha_1\beta_2\\
(\gamma_1 + \gamma_2)s_3 &= \gamma(r_1 + r_2) = \alpha_2\beta_1
\end{aligned}\right\} \tag{29}$$

$$\left.\begin{aligned}
(\alpha_1 - \alpha)h_1 &= 2\alpha r_1 = 2\alpha_1 r = 2\alpha_2 s_2 = 2\alpha_3 s_3\\
(\beta_2 - \beta)h_2 &= 2\beta r_2 = 2\beta_2 r = 2\beta_1 s_1 = 2\beta_3 s_3\\
(\gamma_3 - \gamma)h_3 &= 2\gamma r_3 = 2\gamma_3 r = 2\gamma_1 s_1 = 2\gamma_2 s_2\\
(\alpha_2 + \alpha_3)h_1 &= 2\alpha s = 2\alpha_1 s_1 = 2\alpha_2 r_3 = 2\alpha_3 r_2\\
(\beta_3 + \beta_1)h_2 &= 2\beta s = 2\beta_2 s_2 = 2\beta_1 r_3 = 2\beta_3 r_1\\
(\gamma_1 + \gamma_2)h_3 &= 2\gamma s = 2\gamma_3 s_3 = 2\gamma_1 r_2 = 2\gamma_2 r_1
\end{aligned}\right\} \tag{30}$$

$$\left.\begin{aligned}
(\alpha_1 - \alpha)a &= (\alpha_2 + \alpha_3)(r_1 - r)\\
(\beta_2 - \beta)b &= (\beta_3 + \beta_1)(r_2 - r)\\
(\gamma_3 - \gamma)c &= (\gamma_1 - \gamma_2)(r_3 - r)\\
(\alpha_2 + \alpha_3)a &= (\alpha_1 - \alpha)(r_2 + r_3)\\
(\beta_3 + \beta_1)b &= (\beta_2 - \beta)(r_3 + r_1)\\
(\gamma_1 + \gamma_2)c &= (\gamma_3 - \gamma)(r_1 + r_2)
\end{aligned}\right\} \tag{31}$$

$$\left.\begin{aligned}
(\alpha_1 - \alpha)b &= (\beta_2 - \beta)\gamma_1 = (\beta_3 + \beta_1)\gamma \\
(\alpha_1 - \alpha)c &= (\gamma_3 - \gamma)\beta_1 = (\gamma_1 + \gamma_2)\beta \\
(\beta_2 - \beta)c &= (\gamma_3 - \gamma)\alpha_2 = (\gamma_1 + \gamma_2)\alpha \\
(\beta_2 - \beta)a &= (\alpha_1 - \alpha)\gamma_3 = (\alpha_2 + \alpha_3)\gamma \\
(\gamma_3 - \gamma)a &= (\alpha_1 - \alpha)\beta_3 = (\alpha_2 + \alpha_3)\beta \\
(\gamma_3 - \gamma)b &= (\beta_2 - \beta)\alpha_3 = (\beta_3 + \beta_1)\alpha \\
(\alpha_2 + \alpha_3)b &= (\beta_3 + \beta_1)\gamma_3 = (\beta_2 - \beta)\gamma_3 \\
(\alpha_2 + \alpha_3)c &= (\gamma_1 + \gamma_2)\beta_3 = (\gamma_3 - \gamma)\beta_2 \\
(\beta_3 + \beta_1)c &= (\gamma_1 + \gamma_2)\alpha_3 = (\gamma_3 - \gamma)\alpha_2 \\
(\beta_3 + \beta_1)a &= (\alpha_2 + \alpha_3)\gamma_1 = (\alpha_1 - \alpha)\gamma_3 \\
(\gamma_1 + \gamma_2)a &= (\alpha_3 + \alpha_2)\beta_1 = (\alpha_1 - \alpha)\beta_2 \\
(\gamma_1 + \gamma_2)b &= (\beta_3 + \beta_1)\alpha_2 = (\beta_2 - \beta)\alpha_2
\end{aligned}\right\} \quad (32)$$

$$\begin{aligned}
(\alpha_1 - \alpha)\beta &= (\gamma_3 - \gamma)(r_1 - r) \\
(\alpha_1 - \alpha)\gamma &= (\beta_2 - \beta)(r_1 - r) \\
(\alpha_1 - \alpha)\beta_1 &= (\gamma_1 + \gamma_2)(r_1 - r)
\end{aligned}$$

[redacted]

$$\left.\begin{aligned}
(a_2+a_3)\beta_2 &= (\gamma_1+\gamma_2)(r_2+r_3)\\
(a_2+a_3)\gamma_2 &= (\beta_2-\beta\ )(r_2+r_3)\\
(a_2+a_3)\beta_3 &= (\gamma_3-\gamma\ )(r_2+r_3)\\
(a_2+a_3)\gamma_3 &= (\beta_3+\beta_1)(r_2+r_3)\\[1ex]
(\beta_3+\beta_1)\gamma_1 &= (a_1-a\ )(r_3+r_1)\\
(\beta_3+\beta_1)\,a_1 &= (\gamma_1+\gamma_2)(r_3+r_1)\\
(\beta_3+\beta_1)\,\gamma_3 &= (a_2+a_3)(r_3+r_1)\\
(\beta_3+\beta_1)\,a_3 &= (\gamma_3-\gamma_1)(r_3+r_1)\\[1ex]
(\gamma_1+\gamma_2)\,a_1 &= (\beta_3+\beta_1)(r_1+r_2)\\
(\gamma_1+\gamma_2)\beta_1 &= (a_1-a\ )(r_1+r_2)\\
(\gamma_1+\gamma_2)\,a_2 &= (\beta_2-\beta\ )(r_1+r_2)\\
(\gamma_1+\gamma_2)\beta_2 &= (a_2+a_3)(r_1+r_2)
\end{aligned}\right\}\quad(34)$$

$$\left.\begin{aligned}
(a_1-a\ )r_2 &= (a_2+a_3)s_3 = aa_2 = \beta_1\gamma_2 = \beta_2\gamma\\
(a_1-a\ )r_3 &= (a_2+a_3)s_2 = aa_3 = \beta\ \gamma_3 = \beta_3\gamma_1\\
(\beta_2-\beta\ )r_3 &= (\beta_3+\beta_1)s_1 = b\beta_3 = \gamma_3 a\ = \gamma_2 a_3\\
(\beta_2-\beta\ )r_1 &= (\beta_3+\beta_1)s_3 = b\beta_1 = \gamma\ a_1 = \gamma_1 a_2\\
(\gamma_3-\gamma\ )r_1 &= (\gamma_1+\gamma_2)s_2 = c\,\gamma_1 = a_1\beta\ = a_3\beta_1\\
(\gamma_3-\gamma\ )r_2 &= (\gamma_1+\gamma_2)s_1 = c\ \gamma_2 = a\ \beta_2 = a_2\beta_3\\[1ex]
(a_2+a_3)r\ &= (a_1-a\ )s_1 = aa\ = \beta\ \gamma_2 = \beta_3\gamma\\
(a_2+a_3)r_1 &= (a_1-a\ )s\ = a\,a_1 = \beta_1\gamma_3 = \beta_2\gamma_1\\
(\beta_3+\beta_1)r\ &= (\beta_2-\beta\ )s_2 = b\beta\ = \gamma_1 a\ = \gamma\ a_3\\
(\beta_3+\beta_1)r_2 &= (\beta_2-\beta\ )s\ = b\beta_2 = \gamma_2 a_1 = \gamma_3 a_2\\
(\gamma_1+\gamma_2)r\ &= (\gamma_3-\gamma\ )s_3 = c\,\gamma\ = a\ \beta_1 = a_2\beta\\
(\gamma_1+\gamma_2)r_3 &= (\gamma_3-\gamma\ )s\ = c\ \gamma_3 = a_1\beta_3 = a_3\beta_2
\end{aligned}\right\}\quad(35)$$

$$\left.\begin{aligned}
(\alpha_1-\alpha)^2 &= a^2+(r_1-r)^2=\frac{a^2bcrr_1}{\Delta^2}=\frac{a^2bc}{ss_1}\\
&=(r_1-r)\frac{(r_2+r_3)(r_3+r_1)(r_1+r_2)}{r_2r_3+r_3r_1+r_1r_2}\\
(\beta_2-\beta)^2 &= b^2+(r_2-r)^2=\frac{ab^2crr_2}{\Delta^2}=\frac{ab^2c}{ss_2}\\
&=(r_2-r)\quad[\qquad\qquad\qquad]\\
(\gamma_3-\gamma)^2 &= c^2+(r_3-r)^2=\frac{abc^2rr_3}{\Delta^2}=\frac{abc^2}{ss_3}\\
&=(r_3-r)\quad[\qquad\qquad\qquad]\\
(\alpha_2+\alpha_3)^2 &= a^2+(r_2+r_3)^2=\frac{a^2bcr_2r_3}{\Delta^2}=\frac{a^2bc}{s_2s_3}\\
&=(r_2+r_3)\quad[\qquad\qquad\qquad]\\
(\beta_3+\beta_1)^2 &= b^2+(r_3+r_1)^2=\frac{ab^2cr_3r_1}{\Delta^2}=\frac{ab^2c}{s_3s_1}\\
&=(r_3+r_1)\quad[\qquad\qquad\qquad]\\
(\gamma_1+\gamma_2)^2 &= c^2+(r_1+r_2)^2=\frac{abc^2r_1r_2}{\Delta^2}=\frac{abc^2}{s_1s_2}\\
&=(r_1+r_2)\quad[\qquad\qquad\qquad]
\end{aligned}\right\}\quad(36)$$

[redacted]

$$
\left.
\begin{aligned}
(\alpha_1-\alpha)^2 &= (\beta_3+\beta_1)\beta_1-(\beta_2-\beta)\beta \\
&= (\gamma_1+\gamma_2)\gamma_2-(\gamma_3-\gamma)\gamma \\
(\beta_2-\beta)^2 &= (\gamma_1+\gamma_2)\gamma_2-(\gamma_3-\gamma)\gamma \\
&= (\alpha_2+\alpha_3)\alpha_2-(\alpha_1-\alpha)\alpha \\
(\gamma_3-\gamma)^2 &= (\alpha_2+\alpha_3)\alpha_3-(\alpha_1-\alpha)\alpha \\
&= (\beta_3+\beta_1)\beta_3-(\beta_2-\beta)\beta_1 \\
(\alpha_2+\alpha_3)^2 &= (\beta_3+\beta_1)\beta_3+(\beta_2-\beta)\beta_2 \\
&= (\gamma_1+\gamma_2)\gamma_2+(\gamma_3-\gamma)\gamma_3 \\
(\beta_3+\beta_1)^2 &= (\gamma_1+\gamma_2)\gamma_1+(\gamma_3-\gamma)\gamma_3 \\
&= (\alpha_2+\alpha_3)\alpha_3+(\alpha_1-\alpha)\alpha_1 \\
(\gamma_1+\gamma_2)^2 &= (\alpha_2+\alpha_3)\alpha_2+(\alpha_1-\alpha)\alpha_1 \\
&= (\beta_3+\beta_1)\beta_1+(\beta_2-\beta)\beta_2
\end{aligned}
\right\}
\tag{39}
$$

$$
\left.
\begin{aligned}
(\alpha_1-\alpha)(\alpha_2+\alpha_3) &= (\beta_3+\beta_1)\beta_2-(\beta_2-\beta)\beta_3 \\
&= (\gamma_1+\gamma_2)\gamma_3-(\gamma_3-\gamma_1)\gamma_2 \\
&= (\beta_3+\beta_1)\beta+(\beta_2-\beta)\beta_1 \\
&= (\gamma_1+\gamma_2)\gamma+(\gamma_3-\gamma_1)\gamma_1 \\
(\beta_2-\beta)(\beta_3+\beta_1) &= (\gamma_1+\gamma_2)\gamma_3-(\gamma_3-\gamma)\gamma_1 \\
&= (\alpha_2+\alpha_3)\alpha_1-(\alpha_1-\alpha)\alpha_3 \\
&= (\gamma_1+\gamma_2)\gamma+(\gamma_3-\gamma)\gamma_2 \\
&= (\alpha_2+\alpha_3)\alpha+(\alpha_1-\alpha)\alpha_2 \\
(\gamma_3-\gamma)(\gamma_1+\gamma_2) &= (\alpha_2+\alpha_3)\alpha_1-(\alpha_1-\alpha)\alpha_2 \\
&= (\beta_3+\beta_1)\beta_2-(\beta_2-\beta)\beta_1 \\
&= (\alpha_2+\alpha_3)\alpha+(\alpha_1-\alpha)\alpha_3 \\
&= (\beta_3+\beta_1)\beta+(\beta_2-\beta)\beta_3
\end{aligned}
\right\}
\tag{40}
$$

$$
\left.
\begin{aligned}
(\alpha_1-\alpha)(\beta_2-\beta)(\gamma_3-\gamma) : (\alpha_2+\alpha_3)(\beta_3+\beta_1)(\gamma_1+\gamma_2) &= r : s \\
(\alpha_1-\alpha)(\beta_3+\beta_1)(\gamma_1+\gamma_2) : (\alpha_2+\alpha_3)(\beta_2-\beta)(\gamma_3-\gamma) &= r_1 : s_1 \\
(\alpha_2+\alpha_3)(\beta_2-\beta)(\gamma_1+\gamma_2) : (\alpha_1-\alpha)(\beta_3+\beta_1)(\gamma_3-\gamma) &= r_2 : s_2 \\
(\alpha_2+\alpha_3)(\beta_3+\beta_1)(\gamma_3-\gamma) : (\alpha_1-\alpha)(\beta_2-\beta)(\gamma_1+\gamma_2) &= r_3 : s_3
\end{aligned}
\right\}
\tag{41}
$$

By combining (41) with (8) and (20) other proportions may be obtained which it is needless to write down. T. S. Davies (in the *Ladies' Diary* for 1835, p. 53) gives one of them:

$$(\alpha_1-\alpha)(\beta_2-\beta)(\gamma_3-\gamma) : (\alpha_2+\alpha_3)(\beta_3+\beta_1)(\gamma_1+\gamma_2) = \alpha\beta\gamma : abc \qquad (42)$$

$$\left.\begin{aligned}&(\alpha_2+\alpha_3)(\beta_3+\beta_1)(\gamma_1+\gamma_2)\\ &\quad=(\alpha_2+\alpha_3)(\beta_2-\beta)(\gamma_3-\gamma)+(\alpha_1-\alpha)(\beta_3+\beta_1)(\gamma_3-\gamma)\\ &\qquad\qquad+(\alpha_1-\alpha)(\beta_2-\beta)(\gamma_1+\gamma_2)\end{aligned}\right\}\qquad(43)$$

$$\left.\begin{aligned}&(\alpha_1-\alpha)\alpha s=(\beta_2-\beta)\beta s=(\gamma_3-\gamma)\gamma s\\ &=(\alpha_2+\alpha_3)\alpha r_1=(\beta_3+\beta_1)\beta r_2=(\gamma_1+\gamma_2)\gamma r_3\\ &=(\alpha_1-\alpha)\alpha_1 s_1=(\beta_2-\beta)\beta_1 r_3=(\gamma_3-\gamma)\gamma_1 r_2\\ &=(\alpha_2+\alpha_3)\alpha_1 r=(\beta_3+\beta_1)\beta_1 s_1=(\gamma_1+\gamma_2)\gamma_1 s_1\\ &=(\alpha_1-\alpha)\alpha_2 r_3=(\beta_2-\beta)\beta_2 s_2=(\gamma_3-\gamma)\gamma_2 r_1\\ &=(\alpha_2+\alpha_3)\alpha_2 s_2=(\beta_3+\beta_1)\beta_2 r=(\gamma_1+\gamma_2)\gamma_2 s_2\\ &=(\alpha_1-\alpha)\alpha_3 r_2=(\beta_2-\beta)\beta_3 r_1=(\gamma_3-\gamma)\gamma_3 s_3\\ &=(\alpha_2+\alpha_3)\alpha_3 s_3=(\beta_3+\beta_1)\beta_3 s_3=(\gamma_1+\gamma_2)\gamma_3 r_3\\ &\qquad=abc\end{aligned}\right\}\qquad(44)$$

[redacted]

$$\left.\begin{aligned}
\frac{a_1 - a}{a_1} + \frac{\beta_2 - \beta}{\beta_2} + \frac{\gamma_3 - \gamma}{\gamma_3} &= 2\\
-\frac{a_1 - a}{a} + \frac{\beta_3 + \beta_1}{\beta_3} + \frac{\gamma_1 + \gamma_2}{\gamma_2} &= 2\\
\frac{a_2 + a_3}{a_3} - \frac{\beta_2 - \beta}{\beta} + \frac{\gamma_1 + \gamma_2}{\gamma_1} &= 2\\
\frac{a_2 + a_3}{a_2} + \frac{\beta_3 + \beta_1}{\beta_1} - \frac{\gamma_3 - \gamma}{\gamma} &= 2
\end{aligned}\right\} \qquad (47)$$

The first of these equations is a particular case of a theorem given by Vecten in Gergonne's *Annales*, IX., 277–9 (1819).

$$\left.\begin{aligned}
\frac{1}{a^2} + \frac{1}{a_1^2} + \frac{1}{a_2^2} + \frac{1}{a_3^2} &= \frac{4}{h_1^2}\\
\frac{1}{\beta^2} + \frac{1}{\beta_1^2} + \frac{1}{\beta_2^2} + \frac{1}{\beta_3^2} &= \frac{4}{h_2^2}\\
\frac{1}{\gamma^2} + \frac{1}{\gamma_1^2} + \frac{1}{\gamma_2^2} + \frac{1}{\gamma_3^2} &= \frac{4}{h_3^2}
\end{aligned}\right\} \qquad (48)$$

$$\Sigma\left(\frac{1}{a^2}\right) + \Sigma\left(\frac{1}{\beta^2}\right) + \Sigma\left(\frac{1}{\gamma^2}\right) = \frac{1}{r^2} + \frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} \qquad (49)$$

See (35) of the r formulae.*

$$\left.\begin{aligned}
4\Delta_0 &= 2(a_2 + a_3)\,a_1 = 2(\beta_3 + \beta_1)\,\beta_2 = 2(\gamma_1 + \gamma_2)\,\gamma_3\\
&= (a_1 - a)(a_2 + a_3) + (\beta_2 - \beta)(\beta_3 + \beta_1) + (\gamma_3 - \gamma)(\gamma_1 + \gamma_2)\\
4\Delta_1 &= 2(a_2 + a_3)\,a = 2(\beta_2 - \beta)\,\beta_3 = 2(\gamma_3 - \gamma)\,\gamma_2\\
&= -(a_1 - a)(a_2 + a_3) + (\beta_2 - \beta)(\beta_3 + \beta_1) + (\gamma_3 - \gamma)(\gamma_1 + \gamma_2)\\
4\Delta_2 &= 2(a_1 - a)\,a_3 = 2(\beta_3 + \beta_1)\,\beta = 2(\gamma_3 - \gamma)\,\gamma_1\\
&= (a_1 - a)(a_2 + a_2) - (\beta_2 - \beta)(\beta_3 + \beta_1) + (\gamma_3 - \gamma)(\gamma_1 + \gamma_2)\\
4\Delta_3 &= 2(a_1 - a)\,a_2 = 2(\beta_2 - \beta)\,\beta_1 = 2(\gamma_1 + \gamma_2)\,\gamma\\
&= (a_1 - a)(a_2 + a_3) + (\beta_2 - \beta)(\beta_3 + \beta_1) - (\gamma_3 - \gamma)(\gamma_1 + \gamma_2)
\end{aligned}\right\} \qquad (50)$$

where Δ_0 Δ_1 Δ_2 Δ_3 denote triangles $I_1I_2I_3$ II_3I_2 I_3II_1 I_2I_1I.

* *Proceedings of the Edinburgh Mathematical Society*, Vol. XII., p. 94 (1894).

Historical Notes.

In 1811 the *Ladies' Diary*, which first appeared in 1704, and the *Gentleman's Diary*, which first appeared in 1741, were united and published under the title of the *Lady's and Gentleman's Diary*, which came to an end in 1871. This title will in the notes be shortened to *Diary*.

(1) The values of $\alpha_1\ \beta_2\ \gamma_3$ are given by J. Lowry in the *Ladies' Diary* for for 1836, p. 52; T. S. Davies adds six more in the *Diary* for 1842, p. 79; and Weddle completes the dozen by giving the values of $\alpha\ \beta\ \gamma$ in the *Diary* for 1843, p. 80.

(2) C. J. Matthes in his *Commentatio de Proprietatibus Quinque Circulorum*, pp. 46, 49 (1831).

(3) Weddle in the *Diary* for 1843, p. 81.

(4) Lhuilier in his *Élémens d'Analyse*, p. 215 (1809). The values of $\alpha\alpha_1\ \beta\beta_2\ \gamma\gamma_3$ were however given by J. Lowry in Leybourn's *Mathematical Repository*, old series, I. 394 (1799).

(5) T. T. Wilkinson in *Mathematical Questions from the Educational Times*, XIX. 107 (1873).

(6) C. Adams in *Die merkwürdigsten Eigenschaften des geradlinigen Dreiecks*, p. 36 (1846).

(7) - 12) Weddle in the *Diary* for 1843, pp. 81, 82, 88. The first three proportions of (12) are however implicitly given by Matthes in his *Commentatio*, p. 46 (1831).

(13) The first property was proposed for proof at the *Concours Académique de Clermont*, 1875; the others were given by Mr H. Van Aubel in *Nouvelle Correspondance Mathématique*, IV. 364 (1878).

(24)—(26) Weddle in the *Diary* for 1845, p. 69.

(27)—(29) „ „ „ „ „ „ p. 70.

(30) „ „ „ „ „ „ p. 74.

(31)—(35) „ „ „ „ „ „ p. 71.

(36) The first values of $(a_1 - a)^2$, etc., occur in the *Diary* for 1847, pp. 49-50, in answer to a question proposed the previous year by the editor, W. S. B. Woolhouse. The second values are given by Matthes in his *Commentatio*, pp. 53-4 (1831); the third values by Weddle in the *Diary* for 1845, p. 74. The last values of $(a_2 + a_3)^2$, etc., are given by Franz Unferdinger in Grunert's *Archiv*, XXIX., 436 (1857).

(38) Weddle in the *Diary* for 1843, p. 83.

(39), (40) „ „ „ „ „ 1845, p. 73.

(41) The first proportion is given by Adams in his *Eigenschaften des...Dreiecks*, p. 34 (1846). All four follow at once from eight expressions given by Weddle in the *Diary* for 1843, p. 82.

(43) Weddle in the *Diary* for 1843, p. 82.

(45) „ „ „ „ „ „ p. 83.

(46) „ „ „ „ „ 1845, p. 76.

(47) J. W. Elliott in the *Diary* for 1847, p. 73.

(48) Weddle .. .. 1845, p. 75.

(49) „ .. .. 1845, p. 76.

(50) „ „ „ „ „ „ pp. 72, 75.

§ 10.

Formulae connected with the Angular Bisectors of a Triangle limited at their points of intersection with the sides.

The uniliteral notation for these bisectors

$$l_1 \quad l_2 \quad l_3 \quad \lambda_1 \quad \lambda_2 \quad \lambda_3$$

was suggested by T. S. Davies in the *Lady's and Gentleman's Diary* for 1842, p. 77. In the expressions for them it has been assumed that the sides BC CA AB are in decreasing order of magnitude. Hence it will follow that

BL is less than CL, and BL′ is less than CL′

CM is greater than AM, and CM′ is greater than AM′

AN is less than BN, and AN′ is less than BN′.

The assumption " causes the sign of λ_2 (corresponding to the mean side b) to be contrary to those of λ_1 and λ_3. This must be borne in mind, otherwise the symmetry of the expressions in which these functions (λ_1 λ_2 λ_3) are involved will not be seen." (Weddle in the *Diary* for 1848, p. 76.)

Two fundamental theorems* regarding two sides of a triangle and the bisectors of the angles between them give the following proportions:

$$b : c = u_2 : u_1 = u_2' : u_1'$$
$$bc = u_1 u_2 + l_1^2 = u_1' u_2' - \lambda_1^2$$

Hence are derived

$$\left.\begin{aligned} b^2 &= u_2^2 + l_1^2 \cdot \frac{u_2}{u_1} = u_2'^2 - \lambda_1^2 \cdot \frac{u_2'}{u_1'} \\ c^2 &= u_1^2 + l_1^2 \cdot \frac{u_1}{u_2} = u_1'^2 - \lambda_1^2 \cdot \frac{u_1'}{u_2'} \end{aligned}\right\} \qquad (1)$$

Segments of the sides in terms of the sides

$$\left.\begin{aligned} u_1 &= \frac{ca}{b+c} \quad & v_1 &= \frac{ab}{c+a} \quad & w_1 &= \frac{bc}{a+b} \\ u_1' &= \frac{ca}{b-c} \quad & v_1' &= \frac{ab}{a-c} \quad & w_1' &= \frac{bc}{a-b} \\ u_2 &= \frac{ab}{b+c} \quad & v_2 &= \frac{bc}{c+a} \quad & w_2 &= \frac{ca}{a+b} \end{aligned}\right\} \qquad (2)$$

[redacted]

$$\frac{1}{LL'} - \frac{1}{MM'} + \frac{1}{NN'} = 0 \qquad (4)$$

$$\frac{a^2}{LL'} - \frac{b^2}{MM'} + \frac{c^2}{NN'} = 0 \qquad (5)$$

$$\frac{a}{LL'} - \frac{b}{MM'} + \frac{c}{NN'} = \frac{(b-c)(c-a)(a-b)}{2abc} \qquad (6)$$

The segments of the sides in terms of each other.

$$u_1 = u_1'\frac{u_2' - u_1'}{u_2' + u_1'} \qquad u_2 = u_2'\frac{u_2' - u_1'}{u_2' + u_1'}$$

and so on. (7)

$$u_1' = u_1\frac{u_2 + u_1}{u_2 - u_1} \qquad u_2' = u_2\frac{u_2 + u_1}{u_2 - u_1}$$

and so on.

$$u_1' + u_1 = u_2' - u_2 = LL' = \frac{2u_1u_2}{u_2 - u_1} \qquad (8)$$

and so on.

$$LL'^2 = l_1^2 + \lambda_1^2 \qquad MM'^2 = l_2^2 + \lambda_2^2 \qquad NN'^2 = l_3^2 + \lambda_3^2 \qquad (9)$$

$$\left.\begin{aligned} l_1 &= \frac{2\sqrt{bcss_1}}{(b+c)} = \frac{2\Delta\sqrt{bcrr_1}}{(b+c)rr_1} \\ l_2 &= \frac{2\sqrt{cass_2}}{(c+a)} = \frac{2\Delta\sqrt{carr_2}}{(c+a)rr_2} \\ l_3 &= \frac{2\sqrt{abss_3}}{(a+b)} = \frac{2\Delta\sqrt{abrr_3}}{(a+b)rr_3} \end{aligned}\right\} \qquad (10)$$

$$\left.\begin{aligned}
\lambda_1 &= \frac{2\sqrt{bcs_2s_3}}{(b-c)} = \frac{2\Delta\sqrt{bcr_2r_3}}{(b-c)r_2r_3}\\
\lambda_2 &= \frac{2\sqrt{cas_3s_1}}{(a-c)} = \frac{2\Delta\sqrt{car_3r_1}}{(a-c)r_3r_1}\\
\lambda_3 &= \frac{2\sqrt{abs_1s_2}}{(a-b)} = \frac{2\Delta\sqrt{abr_1r_2}}{(a-b)r_1r_2}
\end{aligned}\right\} \quad (11)$$

$$\left.\begin{aligned}
\frac{l_1^2}{bc} + \frac{a^2}{(b+c)^2} &= 1\\
\frac{l_2^2}{ca} + \frac{b^2}{(c+a)^2} &= 1\\
\frac{l_3^2}{ab} + \frac{c^2}{(a+b)^2} &= 1
\end{aligned}\right\} \quad (12) \qquad
\left.\begin{aligned}
\frac{a^2}{(b-c)^2} - \frac{\lambda_1^2}{bc} &= 1\\
\frac{b^2}{(a-c)^2} - \frac{\lambda_2^2}{ca} &= 1\\
\frac{c^2}{(a-b)^2} - \frac{\lambda_3^2}{ab} &= 1
\end{aligned}\right\} \quad (13)$$

$$\left.\begin{aligned}
l_1^2(b+c)^2 + \lambda_1^2(b-c)^2 &= 4b^2c^2\\
l_2^2(c+a)^2 + \lambda_2^2(a-c)^2 &= 4c^2a^2\\
l_3^2(a+b)^2 + \lambda_3^2(a-b)^2 &= 4a^2b^2
\end{aligned}\right\} \quad (14)$$

$$\frac{l_1^2}{bc}(b+c)^2 + \frac{l_2^2}{ca}(c+a)^2 + \frac{l_3^2}{ab}(a+b)^2 = 4s^2$$

$$\frac{l_1^2}{bc}(b+c)^2 - \frac{\lambda_2^2}{ca}(a-c)^2 - \frac{\lambda_3^2}{ab}(a-b)^2 = 4s_1^2 \qquad (15)$$

[redacted]

$$\left(\frac{1}{b}+\frac{1}{c}\right)\frac{b^2-c^2}{l_2^2l_3^2}+\left(\frac{1}{c}+\frac{1}{a}\right)\frac{c^2-a^2}{l_3^2l_1^2}+\left(\frac{1}{a}+\frac{1}{b}\right)\frac{a^2-b^2}{l_1^2l_2^2}=0 \qquad (18)$$

$$\frac{\frac{1}{b^2}-\frac{1}{c^2}}{s_2s_3}\cdot\frac{b+c}{l_2^2l_3^2}+\frac{\frac{1}{c^2}-\frac{1}{a^2}}{s_3s_1}\cdot\frac{c+a}{l_3^2l_1^2}+\frac{\frac{1}{a^2}-\frac{1}{b^2}}{s_1s_2}\cdot\frac{a+b}{l_1^2l_2^2}=0 \qquad (19)$$

$$\left.\begin{aligned}u_1u_2+v_1v_2+w_1w_2&=abc\left\{\frac{a}{(b+c)^2}+\frac{b}{(c+a)^2}+\frac{c}{(a+b)^2}\right\}\\ u_1'u_2'+v_1'v_2'+w_1'w_2'&=abc\left\{\frac{a}{(b-c)^2}+\frac{b}{(a-c)^2}+\frac{c}{(a-b)^2}\right\}\end{aligned}\right\} \qquad (20)$$

$$\left.\begin{aligned}u_1u_2+v_1v_2+w_1w_2+(l_1^2+l_2^2+l_3^2)&=bc+ca+ab\\ u_1'u_2'+v_1'v_2'+w_1'w_2'-(\lambda_1^2+\lambda_2^2+\lambda_3^2)&=bc+ca+ab\end{aligned}\right\} \qquad (21)$$

$$l_1\alpha+l_2\beta+l_3\gamma=av_1+bw_1+cu_1 \qquad (22)$$

$$\left.\begin{aligned}&l_1(l_1-\alpha)+l_2(l_2-\beta)+l_3(l_3-\gamma)\\ &\quad=(a-v_1)v_2+(b-w_1)w_2+(c-u_1)u_2\end{aligned}\right\} \qquad (23)$$

$$\left.\begin{aligned}&\alpha^2+\beta^2+\gamma^2-\{(l_1-\alpha)^2+(l_2-\beta)^2+(l_3-\gamma)^2\}\\ &\quad=(u_1+v_1+w_1)(u_2+v_2+w_2)-2(u_1v_2+v_1w_2+w_1u_2)\end{aligned}\right\} \qquad (24)$$

$$\left.\begin{aligned}\frac{1}{u_1v_1w_1}&=\left(\frac{1}{b}+\frac{1}{c}\right)\left(\frac{1}{c}+\frac{1}{a}\right)\left(\frac{1}{a}+\frac{1}{b}\right)\\ \frac{1}{u_1v_1'w_1'}&=\left(\frac{1}{b}+\frac{1}{c}\right)\left(\frac{1}{c}-\frac{1}{a}\right)\left(\frac{1}{b}-\frac{1}{a}\right)\\ \frac{1}{u_1'v_1w_1'}&=\left(\frac{1}{c}-\frac{1}{b}\right)\left(\frac{1}{c}+\frac{1}{a}\right)\left(\frac{1}{b}-\frac{1}{a}\right)\\ \frac{1}{u_1'v_1'w_1}&=\left(\frac{1}{c}-\frac{1}{b}\right)\left(\frac{1}{c}-\frac{1}{a}\right)\left(\frac{1}{a}+\frac{1}{b}\right)\end{aligned}\right\} \qquad (25)$$

These may be put into the forms

$$u_1v_1w_1 : abc = abc : (b+c)(c+a)(a+b)$$

and so on; or

$$\mathrm{BL}\cdot\mathrm{CM}\cdot\mathrm{AN} : abc = abc : \mathrm{D_2D_3}\cdot\mathrm{E_3E_1}\cdot\mathrm{F_1F_2}$$

and so on.

$$\left.\begin{aligned}
u_1 v_1 w_1 &= u_2 v_2 w_2 = \frac{4\Delta R r}{h_1 + h_2 + h_3 - r} \\
u_1 v_1' w_1' &= u_2 v_2' w_2' = -\frac{4\Delta R r_1}{h_1 - h_2 - h_3 + r_1} \\
u_1' v_1 w_1' &= u_2' v_2 w_2' = -\frac{4\Delta R r_2}{h_1 - h_2 + h_3 - r_2} \\
u_1' v_1' w_1 &= u_2' v_2' w_2 = \frac{4\Delta R r_3}{-h_1 - h_2 + h_3 + r_3}
\end{aligned}\right\} \quad (26)$$

$$\left.\begin{aligned}
u_1' v_1' w_1' &= u_2' v_2' w_2' = \frac{a^2 b^2 c^2}{(b-c)(a-c)(a-b)} \\
u_1' v_1 w_1 &= u_2' v_2 w_2 = \frac{a^2 b^2 c^2}{(b-c)(c+a)(a+b)} \\
u_1 v_1' w_1 &= u_2 v_2' w_2 = \frac{a^2 b^2 c^2}{(b+c)(a-c)(a+b)} \\
u_1 v_1 w_1' &= u_2 v_2 w_2' = -\frac{a^2 b^2 c^2}{(b+c)(c+a)(a-b)}
\end{aligned}\right\} \quad (27)$$

These may be put into the forms

$$\mathrm{BL}' \cdot \mathrm{CM}' \cdot \mathrm{AN}' : abc = abc : \mathrm{DD}_1 \cdot \mathrm{EE}_2 \cdot \mathrm{FF}_3$$

and so on.

$$\lambda_1\lambda_2\lambda_3 = \frac{32R\Delta}{s(b-c)(a-c)(a-b)} \tag{30}$$

and so on.

$$BL \cdot CM \cdot AN : l_1 l_2 l_3 = R : 2s \tag{31}$$

$$BL' \cdot CM' \cdot AN' : \lambda_1\lambda_2\lambda_3 = R : 2r \tag{32}$$

$$\left.\begin{aligned} &l_1 l_2 l_3 (b+c)(c+a)(a+b) \\ &= 8\alpha\,\beta\,\gamma\,s^3 = 8\alpha_1\beta_1\gamma_1 s s_1^2 = 8\alpha_2\beta_2\gamma_2 s s_2^2 = 8\alpha_3\beta_3\gamma_3 s s_3^2 \\ &= 8\alpha_1\beta_2\gamma_3 s r^2 = 8\alpha\,\beta_3\gamma_2 s r_1^2 = 8\alpha_3\beta\,\gamma_1 s r_2^2 = 8\alpha_2\beta_1\gamma\,s r_3^2 \end{aligned}\right\} \tag{33}$$

$$\left.\begin{aligned} &\lambda_1\lambda_2\lambda_3 (b-c)(a-c)(a-b) \\ &= 8\alpha_1\beta_2\gamma_3 r^3 = 8\alpha\,\beta_3\gamma_2 r r_1^2 = 8\alpha_3\beta\,\gamma_1 r r_2^2 = 8\alpha_2\beta_1\gamma\,r r_3^2 \\ &= 8\alpha\,\beta\,\gamma\,r s^2 = 8\alpha_1\beta_1\gamma_1 r s_1^2 = 8\alpha_2\beta_2\gamma_2 r s_2^2 = 8\alpha_3\beta_3\gamma_3 r s_3^2 \end{aligned}\right\} \tag{34}$$

$$\left.\begin{aligned} l_1 l_2 l_3 \lambda_1\lambda_2\lambda_3 &= \frac{64a^2b^2c^2\Delta^3}{(b^2-c^2)(a^2-c^2)(a^2-b^2)} \\ &= \frac{1024R^2\Delta^5}{(b^2-c^2)(a^2-c^2)(a^2-b^2)} \end{aligned}\right\} \tag{35}$$

$$\frac{2a}{l_1\lambda_1} = \frac{h_3^2 - h_2^2}{h_1 h_2 h_3} \qquad \frac{2b}{l_2\lambda_2} = \frac{h_3^2 - h_1^2}{h_1 h_2 h_3} \qquad \frac{2c}{l_3\lambda_3} = \frac{h_2^2 - h_1^2}{h_1 h_2 h_3} \tag{36}$$

$$\frac{2h_1}{l_1\lambda_1} = \frac{b^2 - c^2}{abc} \qquad \frac{2h_2}{l_2\lambda_2} = \frac{a^2 - c^2}{abc} \qquad \frac{2h_3}{l_3\lambda_3} = \frac{a^2 - b^2}{abc} \tag{37}$$

$$l_1\lambda_1 = \frac{4bc\Delta}{b^2 - c^2} \qquad l_2\lambda_2 = \frac{4ca\Delta}{a^2 - c^2} \qquad l_3\lambda_3 = \frac{4ab\Delta}{a^2 - b^2} \tag{38}$$

$$\frac{a}{l_1\lambda_1} - \frac{b}{l_2\lambda_2} + \frac{c}{l_3\lambda_3} = 0 \tag{39}$$

$$\frac{h_1}{l_1\lambda_1} - \frac{h_2}{l_2\lambda_2} + \frac{h_3}{l_3\lambda_3} = 0 \tag{40}$$

$$\frac{1}{a l_1 \lambda_1} - \frac{1}{b l_2 \lambda_2} + \frac{1}{c l_3 \lambda_3} = 0 \tag{41}$$

$$\frac{1}{h_1 l_1 \lambda_1} - \frac{1}{h_2 l_2 \lambda_2} + \frac{1}{h_3 l_3 \lambda_3} = 0 \tag{42}$$

$$\left.\begin{aligned}
\alpha &: l_1 = r(b+c) : 2\Delta = b+c : 2s \\
\beta &: l_2 = r(c+a) : 2\Delta = c+a : 2s \\
\gamma &: l_3 = r(a+b) : 2\Delta = a+b : 2s \\[4pt]
\alpha_1 &: l_1 = r_1(b+c) : 2\Delta = b+c : 2s_1 \\
\beta_2 &: l_2 = r_2(c+a) : 2\Delta = c+a : 2s_2 \\
\gamma_3 &: l_3 = r_3(a+b) : 2\Delta = a+b : 2s_3 \\[4pt]
\alpha_2 &: \lambda_1 = r_2(b-c) : 2\Delta = b-c : 2s_2 \\
\alpha_3 &: \lambda_1 = r_3(b-c) : 2\Delta = b-c : 2s_3 \\[4pt]
\beta_3 &: \lambda_2 = r_3(a-c) : 2\Delta = a-c : 2s_3 \\
\beta_1 &: \lambda_2 = r_1(a-c) : 2\Delta = a-c : 2s_1 \\[4pt]
\gamma_1 &: \lambda_3 = r_1(a-b) : 2\Delta = a-b : 2s_1 \\
\gamma_2 &: \lambda_3 = r_2(a-b) : 2\Delta = a-b : 2s_2
\end{aligned}\right\} \tag{43}$$

$\alpha : \mathrm{I\,L} = b+c : a$ $\qquad$ $l_1 : \mathrm{I\,L} = 2s : a$

[redacted]

Values such as

$$\left.\begin{array}{l}
l_1 = \dfrac{2\Delta}{r} \cdot \dfrac{a}{D_2D_3} = \dfrac{2\Delta}{r_1} \cdot \dfrac{a_1}{D_2D_3} \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
\lambda_1 = \dfrac{2\Delta}{r_2} \cdot \dfrac{a_2}{DD_1} = \dfrac{2\Delta}{r_3} \cdot \dfrac{a_3}{DD_1} \\
\cdots\cdots\cdots\cdots\cdots\cdots
\end{array}\right\} \quad (46)$$

$$\left.\begin{array}{ll}
IL = \dfrac{a\sqrt{bcrr_1}}{(b+c)r_1} & I_1L = \dfrac{a\sqrt{bcrr_1}}{(b+c)r} \\
\cdots\cdots\cdots\cdots\cdots\cdots & \\
I_2L' = \dfrac{a\sqrt{bcr_2r_3}}{(b-c)r_3} & I_3L' = \dfrac{a\sqrt{bcr_2r_3}}{(b-c)r_2} \\
\cdots\cdots\cdots\cdots\cdots\cdots &
\end{array}\right\} \quad (47)$$

need not be written out at length.

$$\left.\begin{array}{l}
IL \cdot IM \cdot IN = \dfrac{16\Delta R^2r^2}{(b+c)(c+a)(a+b)} \\
I_1L \cdot I_2M \cdot I_3N = \dfrac{16\Delta R^2s^2}{(b+c)(c+a)(a+b)} \\
I_1L \cdot I_1M' \cdot I_1N' = \dfrac{16\Delta R^2r_1^2}{(b+c)(a-c)(a-b)} \\
IL \cdot I_3M' \cdot I_2N' = \dfrac{16\Delta R^2s_1^2}{(b+c)(a-c)(a-b)} \\
I_2L' \cdot I_2M \cdot I_2N' = \dfrac{16\Delta R^2r_2^2}{(b-c)(c+a)(a-b)} \\
I_3L' \cdot IM \cdot I_1N' = \dfrac{16\Delta R^2s_2^2}{(b-c)(c+a)(a-b)} \\
I_3L' \cdot I_3M' \cdot I_3N = \dfrac{16\Delta R^2r_3^2}{(b-c)(a-c)(a+b)} \\
I_2L' \cdot I_1M' \cdot IN = \dfrac{16\Delta R^2s_3^2}{(b-c)(a-c)(a+b)}
\end{array}\right\} \quad (48)$$

$$\left.\begin{aligned} \mathrm{IL}\cdot\mathrm{IM}\cdot\mathrm{IN} &= \frac{4\mathrm{R}r^2}{h_1+h_2+h_3-r} \\ \mathrm{I_1L}\cdot\mathrm{I_2M}\cdot\mathrm{I_3N} &= \frac{4\mathrm{R}rs^2}{h_1+h_2+h_3-r} \end{aligned}\right\} \quad (49)$$

$$\left.\begin{aligned} \alpha\beta\gamma : \mathrm{IL}\cdot\mathrm{IM}\cdot\mathrm{IN} &= (b+c)(c+a)(a+b) : abc \\ &= abc : \mathrm{BL}\cdot\mathrm{CM}\cdot\mathrm{AN} \\ &= h_1+h_2+h_3-r : r \\ &= \alpha_1\beta_2\gamma_3 \cdot \mathrm{I_1L} : \mathrm{I_2M}\cdot\mathrm{I_3N} \end{aligned}\right\} \quad (50)$$

$$\left.\begin{aligned} \mathrm{IL}\cdot\mathrm{IM}\cdot\mathrm{IN} : l_1l_2l_3 &= \mathrm{R}r : 2s^2 \\ \mathrm{I_1L}\cdot\mathrm{I_2M}\cdot\mathrm{I_3N} : l_1l_2l_3 &= \mathrm{R} \;\; : 2r \end{aligned}\right\} \quad (51)$$

$$\left.\begin{aligned} \frac{l_1}{\mathrm{LL'}}\cdot\frac{l_2}{\mathrm{MM'}}\cdot\frac{l_3}{\mathrm{NN'}} &= \frac{(b-c)(a-c)(a-b)}{16\mathrm{R}^2r} = \frac{h_1h_2h_3}{\lambda_1\lambda_2\lambda_3} \\ \frac{\lambda_1}{\mathrm{LL'}}\cdot\frac{\lambda_2}{\mathrm{MM'}}\cdot\frac{\lambda_3}{\mathrm{NN'}} &= \frac{(b+c)(c+a)(a+b)}{16\mathrm{R}^2s} = \frac{h_1h_2h_3}{l_1l_2l_3} \end{aligned}\right\} \quad (52)$$

[redacted]

$$\left.\begin{aligned}
\frac{l_1}{l_2l_3}+\frac{l_1}{\lambda_2\lambda_3}&=\frac{\lambda_1}{\lambda_2l_3}-\frac{\lambda_1}{l_2\lambda_3}=\frac{h_1}{h_2h_3}=\frac{2R}{a^2}\\
\frac{l_2}{l_3l_1}-\frac{l_2}{\lambda_3\lambda_1}&=\frac{\lambda_2}{l_3\lambda_1}+\frac{\lambda_2}{\lambda_3l_1}=\frac{h_2}{h_3h_1}=\frac{2R}{b^2}\\
\frac{l_3}{l_1l_2}+\frac{l_3}{\lambda_1\lambda_2}&=\frac{\lambda_3}{l_1\lambda_2}-\frac{\lambda_3}{\lambda_1l_2}=\frac{h_3}{h_1h_2}=\frac{2R}{c^2}
\end{aligned}\right\}\qquad(55)$$

$$\left.\begin{aligned}
\frac{l_1}{\lambda_1}&=\frac{l_2\lambda_3-\lambda_2l_3}{l_2l_3+\lambda_2\lambda_3}\\
\frac{l_2}{\lambda_2}&=\frac{l_3\lambda_1+\lambda_3l_1}{\lambda_3\lambda_1-l_3l_1}\\
\frac{l_3}{\lambda_3}&=\frac{\lambda_1l_2-l_1\lambda_2}{l_1l_2+\lambda_1\lambda_2}
\end{aligned}\right\}\qquad(56)$$

Weddle remarks that the three preceding relations between $l_1\ l_2\ l_3\ \lambda_1\ \lambda_2\ \lambda_3$ all reduce to

$$l_1l_2l_3=\lambda_1l_2\lambda_3-\lambda_1\lambda_2l_3-l_1\lambda_2\lambda_3\qquad(57)$$

$$\left.\begin{aligned}
l_1&=\frac{2aa_1}{a+a_1} & l_2&=\frac{2\beta\beta_2}{\beta+\beta_2} & l_3&=\frac{2\gamma\gamma_3}{\gamma+\gamma_3}\\
\frac{2}{l_1}&=\frac{1}{a}+\frac{1}{a_1} & \frac{2}{l_2}&=\frac{1}{\beta}+\frac{1}{\beta_2} & \frac{2}{l_3}&=\frac{1}{\gamma}+\frac{1}{\gamma_3}\\
\lambda_1&=\frac{2a_2a_3}{a_2-a_3} & \lambda_2&=\frac{2\beta_1\beta_3}{\beta_1-\beta_3} & \lambda_3&=\frac{2\gamma_1\gamma_2}{\gamma_1-\gamma_2}\\
\frac{2}{\lambda_1}&=\frac{1}{a_3}-\frac{1}{a_2} & \frac{2}{\lambda_2}&=\frac{1}{\beta_3}-\frac{1}{\beta_1} & \frac{2}{\lambda_3}&=\frac{1}{\gamma_2}-\frac{1}{\gamma_1}
\end{aligned}\right\}\qquad(58)$$

$$\left.\begin{aligned}
2AU&=a_1+a & 2BV&=\beta_2+\beta & 2CW&=\gamma_3+\gamma\\
2AU'&=a_2-a_3 & 2BV'&=\beta_1-\beta_3 & 2CW'&=\gamma_1-\gamma_2
\end{aligned}\right\}\qquad(59)$$

$$\left.\begin{aligned}
\mathrm{AU}\,(\alpha_2+\alpha_3) &= 2\mathrm{R}(b+c)\\
\mathrm{BV}\,(\beta_3+\beta_1) &= 2\mathrm{R}(c+a)\\
\mathrm{CW}\,(\gamma_1+\gamma_2) &= 2\mathrm{R}(a+b)\\
\mathrm{AU}'(\alpha_1-\alpha\,) &= 2\mathrm{R}(b-c)\\
\mathrm{BV}'(\beta_2-\beta\,) &= 2\mathrm{R}(a-c)\\
\mathrm{CW}'(\gamma_3-\gamma\,) &= 2\mathrm{R}(a-b)
\end{aligned}\right\}\quad(60)$$

$$\left.\begin{aligned}
\mathrm{AU}\cdot l_1 &= \mathrm{AU}'\cdot\lambda_1 = bc\\
\mathrm{BV}\cdot l_2 &= \mathrm{BV}'\cdot\lambda_2 = ca\\
\mathrm{CW}\cdot l_3 &= \mathrm{CW}'\cdot\lambda_3 = ab
\end{aligned}\right\}\quad(61)$$

$$\left.\begin{aligned}
\mathrm{AU}^2 &= \frac{bc(b+c)^2}{4ss_1} & \mathrm{AU}'^2 &= \frac{bc(b-c)^2}{4s_2s_3}\\
\mathrm{BV}^2 &= \frac{ca(c+a)^2}{4ss_2} & \mathrm{BV}'^2 &= \frac{ac(a-c)^2}{4s_3s_1}\\
\mathrm{CW}^2 &= \frac{ab(a+b)^2}{4ss_3} & \mathrm{CW}'^2 &= \frac{ab(a-b)^2}{4s_1s_2}
\end{aligned}\right\}\quad(62)$$

[redacted]

$$\left.\begin{aligned}
\frac{4}{l_1^2}+\frac{4}{\lambda_1^2}&=\frac{1}{a^2}+\frac{1}{a_1^2}+\frac{1}{a_2^2}+\frac{1}{a_3^2}\\
\frac{4}{l_2^2}+\frac{4}{\lambda_2^2}&=\frac{1}{\beta^2}+\frac{1}{\beta_1^2}+\frac{1}{\beta_2^2}+\frac{1}{\beta_3^2}\\
\frac{4}{l_3^2}+\frac{4}{\lambda_3^2}&=\frac{1}{\gamma^2}+\frac{1}{\gamma_1^2}+\frac{1}{\gamma_2^2}+\frac{1}{\gamma_3^2}
\end{aligned}\right\}\quad(66)$$

$$\left.\begin{aligned}
\frac{a}{l_1}+\frac{\beta}{l_2}+\frac{\gamma}{l_3}&=2\\
\frac{a_1}{l_1}-\frac{\beta_1}{\lambda_2}+\frac{\gamma_1}{\lambda_3}&=2\\
\frac{\beta_2}{l_2}+\frac{\gamma_2}{\lambda_3}-\frac{a_2}{\lambda_1}&=2\\
\frac{\gamma_3}{l_3}+\frac{a_3}{\lambda_1}+\frac{\beta_3}{\lambda_2}&=2
\end{aligned}\right\}\quad(67)
\qquad
\left.\begin{aligned}
\frac{a_1}{\lambda_1}-\frac{\beta_2}{\lambda_2}+\frac{\gamma_3}{\lambda_3}&=0\\
\frac{a}{\lambda_1}+\frac{\beta_3}{l_2}-\frac{\gamma_2}{l_3}&=0\\
\frac{a_3}{l_1}+\frac{\beta}{\lambda_2}-\frac{\gamma_1}{l_3}&=0\\
\frac{a_2}{l_1}-\frac{\beta_1}{l_2}+\frac{\gamma}{\lambda_3}&=0
\end{aligned}\right\}\quad(68)$$

Let AI BI CI meet MN NL LM respectively at L_1 M_1 N_1

$$\left.\begin{aligned}
AL_1 : IL_1 &= AL : IL = h_1 : r\\
BM_1 : IM_1 &= BM : IM = h_2 : r\\
CN_1 : IN_1 &= CN : IN = h_3 : r
\end{aligned}\right\}\quad(69)$$

$$\left.\begin{aligned}
AL_1&=\frac{h_1 a}{h_1+r} & BM_1&=\frac{h_2\beta}{h_2+r} & CN_1&=\frac{h_3\gamma}{h_3+r}\\
IL_1&=\frac{ra}{h_1+r} & IM_1&=\frac{r\beta}{h_2+r} & IN_1&=\frac{r\gamma}{h_3+r}
\end{aligned}\right\}\quad(70)$$

Matthes (p. 47) gives the values

$$\left.\begin{aligned}
AL_1&=\frac{h_1\sqrt{bcrr_1}}{(h_1+r)r_1},\ \text{etc.,}\\
IL_1&=\frac{r\sqrt{bcrr_1}}{(h_1+r)r_1},\ \text{etc.}
\end{aligned}\right\}\quad(71)$$

Expressions for the sides of $\triangle LMN$.

$$\left.\begin{aligned} MN^2 &= \frac{abc}{(c+a)^2(a+b)^2} \times \\ &(b^2c + bc^2 - c^2a + ca^2 + a^2b - ab^2 + a^3 - b^3 - c^3 + 3abc) \end{aligned}\right\} \quad (72)$$

NL^2 and LM^2 can be obtained by cyclical permutations of the letters $a\ b\ c$.

These expressions can be put into shorter forms, by help of Landen's theorem that

$$I_1O^2 = R^2 + 2Rr_1$$

For $$4\triangle(R + 2r_1) = 4\triangle R + \frac{16\triangle^2}{2s_1}$$

$$= abc + (a+b+c)(a-b+c)(a+b-c)$$

$$= b^2c + bc^2 - c^2a + ca^2 + a^2b - ab^2 + a^3 - b^3 - c^3 + 3abc$$

Hence

$$MN = \frac{4\triangle \cdot I_1O}{(c+a)(a+b)} \qquad NL = \frac{4\triangle \cdot I_2O}{(a+b)(b+c)} \qquad LM = \frac{4\triangle \cdot I_3O}{(b+c)(c+a)} \quad (73)$$

Matthes (p. 45) in transforming the ten-term factor which occurs in the expression for MN^2 does not appear to have observed the simplification that would result from introducing $R + 2r_1$. He [redacted]

These expressions can be put into shorter forms, by help of Chapple's theorem that

$$IO^2 = R^2 - 2Rr$$

For $$4\Delta(R - 2r) = 4\Delta R - \frac{16\Delta^2}{2s}$$

$$= abc + (-a + b + c)(a - b + c)(a + b - c)$$

$$= -b^2c - bc^2 - c^2a - ca^2 - a^2b - ab^2 + a^3 + b^3 + c^3 + 3abc$$

Hence

$$M'N' = \frac{4\Delta \cdot IO}{(a-c)(a-b)} \quad N'L' = \frac{4\Delta \cdot IO}{(a-b)(b-c)} \quad L'M' = \frac{4\Delta \cdot IO}{(b-c)(a-c)} \quad (75)$$

In deducing the expressions for $M'N'$ $N'L'$ $L'M'$ it has been assumed that a b c are in descending order of magnitude. If the figure do not correspond to this supposition, care must be taken in verifying the equation

$$L'M' = M'N' + N'L'$$

to affix the proper signs to the values of these magnitudes.

HISTORICAL NOTES.

(3) Crelle's *Eigenschaften des...Dreiecks*, p. 39 (1816). The property is probably much older.

(4) Weddle in the *Diary* for 1843, p. 75.

(10) The first values of l_1 l_2 l_3 are given by Vecten in Gergonne's *Annales*, IX., 304 (1818-9); the second values by Matthes in his *Commentatio*, p. 42 (1831).

(11) The first values are given by Weddle in the *Diary* for 1848, p. 78; the second values by Matthes in his *Commentatio*, p. 58 (1831).

(12) Mr Robert E. Anderson.

(14) The first equality is given by Mr Launoy in Bourget's *Journal de Mathématiques Élémentaires*, III. 160 (1879).

(15) The first equality is given in J. A. Grunert's article "Dreieck" in *Supplemente zu Klügel's Wörterbuche der reinen Mathematik*, I. 709 (1833). In this article Grunert gives also (20).

(16)—(19) Mr Robert E. Anderson.

(21) First part in Jacobi's *De Triangulorum Proprietatibus*, p. 8 (1825). Both parts certainly much older.

(22)—(24) Jacobi, p. 13 (1825).

(25) Jacobi, p. 12 (1825), gives the first equality in the first alternative form.

(26) First equality given by Matthes in his *Commentatio*, p. 42 (1831).

(27) First equality given by Marsano in his *Considerazioni sul Triangolo Rettilineo*, p. 29 (1863).

(28) The value of $l_1 l_2 l_3$ is given by Vecten in Gergonne's *Annales*, IX. 304 (1819); that of $\lambda_1 \lambda_2 \lambda_3$ by Weddle in the *Diary* for 1848, p. 78.

(29) The first value is given by Vecten in Gergonne's *Annales*, IX. 305 (1819).

(31), (32) J. W. Elliott in the *Diary* for 1851, p. 58.

(33) The first of these eight values is given by Jacobi, p. 10 (1825).

(35) Weddle in the *Diary* for 1848, p. 78.

(36) ,, ,, ,, ,, ,, ,, p. 81.

(37) ,, ,, ,, ,, ,, ,, p. 80.

(38) *Nouvelles Annales*, 2nd series, IX. 548 (1870).

(39)—(42) Weddle in the *Diary* for 1848, pp. 81-2.

(43)—(45) Matthes, pp. 46, 48, 50 (1831), gives several of these proportions, but they must all have been known long previously.

(46) Matthes, p. 58 (1831), gives the values of $\lambda_1\ \lambda_2\ \lambda_3$, but he does not seem to have observed the corresponding ones for $l_1\ l_2\ l_3$.

(47) Matthes, pp. 48, 51, gives the first half of these values, the first two of (48), and the first of (49).

Formulae connected with the Radii of the Incircle and the Excircles of a Triangle.

By J. S. Mackay, M.A., LL.D.

The following formulae may be added to the list given in the *Proceedings of the Edinburgh Mathematical Society*, Vol. XII., pp. 86–102 (1894).

$$\left.\begin{aligned} a &= \frac{r_1(r_2+r_3)}{s} = \frac{r(r_2+r_3)}{s_1} \\ &= \frac{s^2(r+r_3)+r_3^2(r_1-r_2)}{2sr_3} = \frac{s_1^2(r_1-r_2)+r_2^2(r+r_3)}{2s_1r_2} \end{aligned}\right\} \quad (81)$$

and so on.

$$b+c = \frac{s(r_1+r)}{r_1} = \frac{s_1(r_1+r)}{r}, \;\ldots\ldots \quad (82)$$

$$b-c = \frac{r_1(r_2-r_3)}{s} = \frac{r(r_2-r_3)}{s_1}, \;\ldots\ldots \quad (83)$$

$$\left.\begin{aligned} (b+c)(c+a)(a+b) : abc &= h_1+h_2+h_3-r : r \\ (b+c)(a-c)(a-b) : abc &= h_1-h_2-h_3+r_1 : r_1 \end{aligned}\right\} \quad (84)$$

$$\left.\begin{aligned} &4r^2(r_1^2+r_2^2+r_3^2)+4(r_2^2r_3^2+r_3^2r_1^2+r_1^2r_2^2) \\ &= 4\,\Delta^4\left(\frac{1}{r_2^2r_3^2}+\frac{1}{r_3^2r_1^2}+\frac{1}{r_1^2r_2^2}+\frac{1}{r^2r_1^2}+\frac{1}{r^2r_2^2}+\frac{1}{r^2r_3^2}\right) \\ &= (a^2+b^2+c^2)^2-8\Delta^2 \end{aligned}\right\} \quad (85)$$

$$\Delta^4\left(\frac{1}{r^2}+\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}\right)^2 = (a^2+b^2+c^2)^2 \quad (86)$$

$$2\Delta^4\left(\frac{1}{r^4}+\frac{1}{r_1^4}+\frac{1}{r_2^4}+\frac{1}{r_3^4}\right) = (a^2+b^2+c^2)^2+8\Delta^2 \quad (87)$$

$$\left.\begin{array}{l} r : r_1 = h_1 - 2r : h_1 = h_1 : h_1 + 2r_1 \\ r : r_2 = h_2 - 2r : h_2 = h_2 : h_2 + 2r_2 \\ r : r_3 = h_3 - 2r : h_3 = h_3 : h_3 + 2r_3 \end{array}\right\} \qquad (88)$$

$$\left.\begin{array}{l} r^2 : r_1^2 = h_1 - 2r : h_1 + 2r_1 \\ r^2 : r_2^2 = h_2 - 2r : h_2 + 2r_2 \\ r^2 : r_3^2 = h_3 - 2r : h_3 + 2r_3 \end{array}\right\} \qquad (89)$$

$$\frac{(h_1 - 2r)(h_2 - 2r)(h_3 - 2r)}{(h_1 + 2r_1)(h_2 + 2r_2)(h_3 + 2r_3)} = \frac{r^4}{s^4} \qquad (90)$$

[redacted]

On the real Common Chords of a Point Circle and Ellipse.

By Robert Frederick Davis, M.A.

(1) If O be a given point in the plane of a given conic, the mutual relationship between point and conic is marked, first and foremost, by the existence of a certain determinate straight line (which is always real) known as the polar of O with respect to the conic. Next following the polar in natural order of sequence, come a certain pair of determinate straight lines:—

The single real pair of common chords of the conic and a point circle at O.

(2) As in the generality of cases, it will be best to rely upon analysis for discovery of facts, and then to look to geometry for elucidation.

(3) Consider the conic represented by the equation

$$x^2 + y^2 = (ax + by + c)(a'x + b'y + c') \quad \dots \quad \dots \quad \text{(A)}$$

which really involves five constants, for either c or c' may be put $=1$. It represents the locus of a point which moves in such a manner that the square of its distance from the origin varies as the product of its perpendicular distances upon the fixed straight lines

$$\left.\begin{array}{l} ax + by + c = 0 \\ a'x + b'y + c' = 0 \end{array}\right\} \quad \dots \quad \dots \quad \dots \quad \text{(B)}$$

(4) These two straight lines (B) must be regarded as the real common chords of the conic and a point circle at O. They cannot intersect the conic in real points, and consequently lie outside the conic. Since any circle theoretically intersects a conic in four points lying two and two upon three pairs of common chords, and (A) may be written in the form

$$y^2 - \iota^2 x^2 = (ax + by + c)(a'x + b'y + c'), \quad (\iota^2 = -1)$$

therefore $y \pm \iota x$ is one pair of imaginary chords; and of the other two pairs, one pair must be real and one imaginary.

(5) It may here be noted that the straight lines (B) are equally inclined to either axis of the conic following the general property of the chords of intersection of a circle and conic. For the coefficient of xy vanishes in (A) when $ab' + b'a = 0$; so that perhaps a more convenient form might therefore be taken as

$$x^2 + y^2 = (\lambda x + \mu y + \nu)(\lambda x - \mu y + \nu')$$

Again, when (B) represents coincident straight lines, the origin becomes a focus and the coincident common chords become the corresponding directrix.

(6) The geometrical connexion of the foregoing is as follows:— If the conic be reciprocated with respect to any point O we get a second conic. The foci of the first reciprocate into certain straight lines $\Delta\Delta'$, $\delta\delta'$; while the property that the product of the perpendiculars from the foci upon any tangent to the first conic is constant reciprocates into the property that the (distance OP)^2 of any point on the second conic from O varies as the product of the perpendiculars from P on $\Delta\Delta'$, $\delta\delta'$. (By using Salmon's Theorem.) Thus from Art. (3) we see that $\Delta\Delta'$, $\delta\delta$ are the real common chords of intersection of a point circle at O with the second conic.

(7) It will be convenient to designate the above pair of straight lines the Delta lines of the conic corresponding to O.

(8) Properties of the Delta lines are therefore easily found by [redacted]

$\omega\omega'$ is real, and is found by drawing the tangent at P parallel to UU′ and producing CP to meet UU′ in T.

Also ωT (or $=\omega'$T) is given by the equation

$$\frac{CT^2}{CP^2}+\frac{\omega T^2}{CD^2}=1$$

where since CT > CP ωT^2 is negative. Through T draw TO $\perp^r$ to UU′ (*i.e.*, parallel to the normal at P). Then if a point circle at O pass through ω, ω'

$$OT^2+\omega T^2=0,$$

hence
$$\frac{CT^2}{CP^2}-\frac{OT^2}{CD^2}=1.$$

Hence O lies on the concentric hyperbola passing through P whose conjugate diameter is $=$ CD and is perpendicular to CD. This is obviously the confocal hyperbola through P (for at their point of intersection CP is a common semi-diameter, and the conjugate semi-diameter $=(SP\,.\,SP)^{\frac{1}{2}}$ in each case though in perpendicular directions).

(11) Since in Fig. 25 O may lie on either side of UU′, we see that to any line outside an ellipse correspond two determinate points the point-circles at which have the given line as their common chord with the ellipse.

(12) To determine therefore the Delta lines corresponding to any point O with respect to a given ellipse, we have the following construction. Draw the confocal hyperbola through O intersecting the ellipse at the extremities of the equi-diameters PCP′, pCp'; draw OT parallel to the normal at P to meet CP produced in T; and through T draw UU′ perpendicular to OT or parallel to the tangent at P. Then UU′ is one of the Delta lines required: and a similar construction gives the other.

(13) If upon CP, Cp points Q, q be taken respectively such that $CQ\,.\,CT=CP^2=Cp^2=Cq\,.\,Ct$, then Q, q are obviously the points in which the tangent at O to the confocal hyperbola meets CP, Cp respectively. Consequently Q, O, q are in a straight line; and UU′, uu' intersect on the polar of O. See Art. (9).

(14) Various geometrical properties may be noted. By reciprocation we find that if R be any point on $\Delta\Delta'$ (see Fig. 26*a*), and $R\Sigma$ intersect the curve in Q, Q', then OQ, OQ' each divide the angle ΣOR into parts whose sines are in the ratio $e:1$, and consequently the range $\{RQ\Sigma Q'\}$ is harmonic, as we should expect. This interesting result may be otherwise stated: If BC be the fixed base of a triangle whose variable vertex P describes a straight line, the locus of Q taken on PC such that $\sin QBC : \sin QBP$ is constant is a conic.

(15) Again (in Fig. 26*b*), if the tangent at Q meet $\Delta\Delta'$ in Z, and $Q\Sigma$ meet $\Delta\Delta'$ in R (and the curve in Q'), ZOR is a right angle. Whence ZQ' is the tangent at Q'; and the conjugate points Z, R subtend a right angle at O. Thus given a straight line outside an ellipse, the circle described upon any pair of conjugate points lying on that straight line passes through two fixed points—the point circles at which have the given straight line as their chord of intersection with the ellipse.

(16) The pedal of an ellipse with respect to O is found by reciprocation to be the locus of a point whose distances from three points (one of which is O and the others the feet of the perpendiculars from O on the Delta lines) are connected by a linear relation (bicircular quartic).

[redacted]

(19) To the best of my belief the late Professor Wolstenholme, in his monumental collection of problems (containing as it does such a vast store of interesting matter relating to Conics), does not indicate any construction for the Delta lines, or specify their equations.

Dr. James Booth was evidently aware of their existence, and, in the special case only in which O lies on the axis, employs their properties in his "New Geometrical Methods," Vol. I

Dr. Taylor, in his "Geometry of Conics," mentions their existence, but gives no construction.

I conclude with the following analytical notes.

I. To find the value of λ for which the equation

$$(x-\alpha)^2+(y-\beta)^2-\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)=0 \quad \ldots \quad \ldots \quad \text{(A)}$$

represents straight lines; in other words, to find the common chords of the ellipse and a point-circle at α, β.

The discriminant must $=0$, or

$$\left(1-\frac{\lambda}{a^2}\right)\left(1-\frac{\lambda}{b^2}\right)(\alpha^2+\beta^2+\lambda)-\left(1-\frac{\lambda}{a^2}\right)\beta^2-\left(1-\frac{\lambda}{b^2}\right)\alpha^2=0.$$

It will be found that one value of $\lambda=0$, as we should *a priori* expect since the lines $(y-\beta)=\pm i(x-d)$ form one pair of common chords.

The other two values of λ are given by the equation

$$\lambda^2+\lambda(\alpha^2+\beta^2-a^2-b^2)-(b^2\alpha^2+a^2\beta^2-a^2b^2)=0. \quad \ldots \quad \text{(B)}$$

Note if α, β lies on the ellipse, another value of $\lambda=0$, and the third value of $\lambda=a^2+b^2-\alpha^2-\beta^2$, and the real common chords are

$$(x-\alpha^2)+(y-\beta)^2=(a^2+b^2-\alpha^2-\beta^2)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)$$

which may be thrown into the form

$$(a^2-b^2)\left\{\frac{x\alpha}{a^2}+\frac{y\beta}{b^2}-1\right\}\left\{\frac{x\alpha}{a^2}-\frac{y\beta}{b^2}-\frac{a^2+b^2}{a^2-b^2}\right\}=0$$

the factors representing the tangent at (α,β) and the Frégier line corresponding to α, β.

Again, if α, β lies on the orthocyclic circle the two values of λ are $= \pm (b^2\alpha^2 + a^2\beta^2 - a^2b^2)$

From (A) the lines will be real or not, according as the parallels through the origin are real or not: that is, according as

$$\left(1 - \frac{\lambda}{a^2}\right)x^2 + \left(1 - \frac{\lambda}{b^2}\right)y^2 = 0$$

has real or imaginary roots.

Since this may be written

$$y^2 = \frac{a^2 - \lambda}{\lambda - b^2} \cdot \frac{b^2}{a^2} x^2$$

we see that λ must lie between b^2 and a^2.

Now in (B) if we substitute $\lambda = a^2$ and $\lambda = b^2$ we get $(a^2 - b^2)\alpha^2$ and $(b^2 - a^2)\beta^2$ respectively, the first of which is positive and the second negative. This shows that (B) has always a real root between b^2 and a^2.

II. If P be a point on the ellipse whose excentric angle is α, the coordinates of P are $a\cos\alpha, b\sin\alpha$. It can be easily verified that the semi-axes of the confocal hyperbola through P are

$$c\cos\alpha, c\sin\alpha \quad \text{where} \quad c^2 = a^2 - b^2.$$

For $\frac{a^2\cos^2\alpha}{c^2\cos^2\alpha} - \frac{b^2\sin^2\alpha}{c^2\sin^2\alpha} = 1$, and $c^2\cos^2\alpha + c^2\sin^2\alpha = c^2$.

[redacted]

This shows that the real chords of the point-circle at the point whose coordinates are $c\cos\alpha\sec\phi$ $c\sin\alpha\tan\phi$ are the straight lines

$$\left(\frac{c}{a}x\cos\alpha - a\sec\phi\right) \pm \left(\frac{c}{b}y\sin\alpha + b\tan\phi\right),$$

which expressions are probably new.

Note on Triangle Transformations.

By R. F. Muirhead, M.A.

The investigation given in the following Note was suggested by a passage in the paper by Mr Lemoine, presented to the Society by Dr. Mackay at a recent meeting. The main subject of that paper is what he terms the "*Transformation continue dans le triangle et dans le tétraèdre*"; for the explanation of that phrase and other terms connected with it, the reader is referred to the paper just mentioned. In the notation for the quantities connected with the triangle, however, I shall follow Dr. Mackay's system as explained in his paper in Vol. I. of our *Proceedings*.

I shall use, moreover, the letters α, β, γ provisionally as symbols of operation, to denote what Mr Lemoine calls "*la transformation continue en* A, *en* B, *en* C" respectively. Thus, as shown in Lemoine's paper,

$$\alpha a = a, \ \alpha b = -b, \ \alpha \mathrm{A} = -\mathrm{A}, \ \alpha \mathrm{B} = \pi - \mathrm{B}, \text{ etc.}$$

A compound operation such as $\beta\alpha . \mathrm{A}$ I shall use to mean $\beta(\alpha\mathrm{A})$, and $\dfrac{1}{\alpha}$ to denote the reverse of the operation α. Thus

[redacted]

He states that all these are found to occur, but that he has not yet found any case of the following type:

$$(5)\quad \alpha = 1, \quad \beta, \gamma, 1 \quad \text{all different.}$$

The object of this Note is to account for the non-occurrence of such cases. We may complete the list of typical cases by

$$(6)\quad \alpha \neq 1, \quad \beta = \gamma = 1$$
$$(7)\quad \alpha \neq 1, \quad \beta = \gamma \neq 1.$$

We have
$$\gamma(a) = -a$$
$$\beta\gamma(a) = \beta(-a) = a$$
$$\alpha\beta\gamma(a) = \alpha(a) = a$$

Similarly $\qquad \alpha\beta\gamma(r) = -r, \quad \alpha\beta\gamma(r_1) = -r_1, \quad$ etc.

In fact, if F denote any function whatever, we have

$$\begin{aligned}\alpha\beta\gamma\{F(a, b, c, s, s_1, s_2, s_3, r, r_1, r_2, r_3, h_1, h_2, h_3, \Delta, R)\} \\ = \left.\begin{aligned}\{F(a, b, c, s, s_1, s_2, s_3, -r, -r_1, -r_2, -r_3, \\ -h_1, -h_2, -h_3, -\Delta, -R)\}\end{aligned}\right\}\end{aligned}$$

Thus the compound operator $\alpha\beta\gamma$ changes the signs of certain letters, but leaves them otherwise unaltered.

Now if any identity connecting the functions of the general triangle be written in the form $F = 0$, it is clear that the identity will not be altered by applying the operation $\alpha\beta\gamma$, so long as F consists of terms which, with respect to the letters $r, r_1, \cdots\cdot R$ (whose signs are changed thereby) are either all of *even* dimensions, or all of *odd* dimensions. Such terms will hereafter be referred to, for brevity's sake, as *even*, or *odd* simply. The letters $ab \cdots\cdot s_3$ whose signs are unchanged by the operation $\alpha\beta\gamma$ are not to be reckoned in this connection. Omitting for the present, then, all reference to identities in which F is a *mixed* function, we may say that $\alpha\beta\gamma = 1$, *i.e.* the operator $\alpha\beta\gamma$ reproduces the same identity as we start with.

Now it is easily proved that $\alpha^2 F = F$, whatever the function F may be, or, as we may write it $\alpha^2 = 1$, whence $\alpha = \dfrac{1}{\alpha}$. And the same is true as applied to an identical equation.

But since $\alpha\beta\gamma = 1$, restricting ourselves to identities which are not *mixed*, we have $\alpha \,.\, \alpha\beta\gamma = \alpha$

$$\therefore \quad \alpha^2\beta\gamma = \alpha \qquad\qquad \therefore \quad \beta\gamma = \alpha$$

Hence if $\alpha = 1$, $\beta\gamma = 1$ $\qquad \therefore \quad \beta = \dfrac{1}{\gamma} = \gamma$

This shows that case (5) cannot occur.

Conversely if $\beta = \gamma$, we have $\beta\gamma = \gamma^2 = 1$ $\qquad \therefore \quad \alpha = 1$

This shows that cases (6) and (7) cannot occur.

We may sum up these results by saying that if $\alpha = 1$, then $\beta = \gamma$; and *vice versa*.

Now it may be asked, Are there not *mixed* identities which give results of the types (5), (6), (7)? The answer is that there are; but they are always composite identities, which may be reduced to two or more simpler ones. In fact, if any identity be expressed by $F = 0$ when $F = F_1 + F_2$, F_1 being an *odd*, and F_2 an *even* function, as explained above, then applying $\alpha\beta\gamma$ to the identity $F_1 + F_2 = 0$ we get a new identity $\qquad -F_1 + F_2 = 0$

whence $F_1 = 0$ and $F_2 = 0$ identically; *i.e.* $F = 0$ is composed by adding together two identities $F_1 = 0$ and $F_2 = 0$. It is easy to manufacture such cases, *e.g.*,

$$sr - s_1r_1 + rr_1 - s_2s_3 = 0.$$

This is an identity which belongs to the type (5).

I have in the foregoing omitted all consideration of the angles A, B, C, for the reason that they introduce further complication. [redacted]

Fourth Meeting, February 8th, 1895.

J. S. Mackay, Esq., M.A., LL.D., F.R.S.E., in the Chair.

Theorems in the Products of Related Quantities.

By F. H. Jackson, M.A.

§ 1. The object of this Note is to prove the following theorems:

$$\left.\begin{aligned}(a+b)_{-n} = a_{-n} - na_{-n-1}b_1 + \frac{n\cdot n+1}{2\,!}a_{-n-2}b_2 \\ - \frac{n\cdot n+1\cdot n+2}{3\,!}a_{-n-3}b_3 + \cdots\cdots\end{aligned}\right\} \quad (1)$$

in which $a_{-n} = \dfrac{1}{a+n\cdot a+n-1\cdots\cdots a+1\cdot}$ and the series is subject to conditions for convergence.

$$\left.\begin{aligned}\frac{P(0)}{x+n} - {}_nC_1\frac{P(y)}{x+n-1} + {}_nC_2\frac{P(2y)}{x+n-2} - \cdots\cdots \text{ to } n+1 \text{ terms} \\ \equiv(-1)^n\frac{x\,!\,n\,!}{x+n\,!}\cdot\frac{P(xy+ny)}{x}\end{aligned}\right\} \quad (2)$$

$$P(0) - {}_nC_1P(y) + {}_nC_2P(2y) - \cdots\cdots \text{ to } n+1 \text{ terms} \quad \equiv 0 \qquad (3)$$

$$N(0) - {}_nC_1N(y) + {}_nC_2N(2y) - \cdots\cdots \quad ,, \quad ,, \quad ,, \quad \equiv(-y)^n.\,n\,! \quad (4)$$

in which $P(y) \equiv (a+y)(b+y)(c+y) \cdots\cdots$ to p factors

$N(y) \equiv (a+y)(b+y)(c+y) \ldots\ldots$ to n factors

and the quantities $abcd \cdots\cdots x$ are unrestricted. In theorem (2) $p \gtrless n$, but in (3) $p < n$

The following theorems are derived from the above

$$\left.\begin{aligned}\frac{(a)_p}{x} - {}_nC_1\frac{(a-1)_p}{x-1} + {}_nC_2\frac{(a-2)_p}{x+2} - \cdots\cdots + (-1)^n\frac{(a-n)_p}{x+n} \\ \equiv \frac{x!\,n!}{x+n!}\;\frac{(x+a)_p}{x}\end{aligned}\right\} \quad (5)$$

$$\frac{(a)_p}{x} - {}_nC_1\frac{(a+1)_p}{x-1} + \cdots\cdots \equiv (-1)^n\frac{x-n!\,n!}{x!}\,\frac{(x+a)_p}{x-n} \quad (6)$$

$$(p \lesseqgtr n)$$

$$(x)_n - {}_nC_1(x+y)_n + \cdots + (-1)^n(x+ny)_n \equiv (-y)^n n! \quad (7)$$

$$(x)_p - {}_nC_1(x+y)_p + \cdots + (-1)^n(x+ny)_p \equiv 0 \qquad (p<n) \quad (8)$$

These correspond in form with the following theorems in the products of equal quantities (powers)

$$\frac{(a)^p}{x} - {}_nC_1\frac{(a-1)^p}{x+1} + \cdots + (-1)_n\frac{(a-n)^p}{x+n} \equiv \frac{x!\,n!}{x+n!}\,\frac{(x+a)^p}{x} \quad (9)$$

and three others formed from (6) (7) and (8) by changing subscript letters to indices, thus $(x+y)_n$ to $(x+y)^n$.

§2. The theorems in powers corresponding to (7) and (8) are [redacted]

VANDERMONDE'S THEOREM.

§ 3. Can any meaning be attached to the theorem

$$(a+b)_n = a_n + n_1 a_{n-1} b_1 + \frac{n_2}{2\,!} a_{n-2} b_2 + \cdots$$

when n is not restricted, as hitherto, to being a positive integer? In Vandermonde's Theorem a_n represents the product of n related factors $a \cdot a-1 \cdot a-2 \cdots\cdot a-n+1$, and certainly so long as we regard a_n as the product of n factors such expressions as

$$a_{-n}\,, \quad a_{\frac{1}{2}}\,, \quad a_{\frac{p}{q}}\,,$$

seem beyond our comprehension. Exactly the same might have been written of the quantities

$$a^{-n}, \quad a^{\frac{p}{q}}, \quad a^{\frac{1}{2}},$$

so long as a^m was regarded as the product of m factors each equal to a. The Binomial Theorem, until fractional and negative indices were interpreted, was a finite algebraical identity; but as soon as a fundamental law $a^m \times a^n = a^{m+n}$, was assumed in the Theory of Indices, then the expressions

$$a^{-m}, \quad a^{\frac{p}{q}},$$

were interpreted, and the Binomial Theorem was shown to hold (with certain restrictions) for fractional and negative powers.

§ 4. Now in the expressions a_m, a_n, (the usual meanings being attached) we have these relations

$$a_n \times (a-n)_m \quad = a_{m+n} \qquad (\alpha)$$

$$a_m \times (a-m)_n \quad = a_{m+n} \qquad (\beta)$$

$$a_n \times (a-n)_{m-n} = a_m \qquad (\gamma)$$

These are all expressions of one law. Let us assume this law as general and interpret a_{-n}, a_0, in accordance with our assumption.

In the relation (α) put $n=0$ then we have $a_0 \times a_m = a_m$ whence $a_0 = 1$ and this is analogous to $a^0 = 1$.

In the relation (γ) change n to $-r$ then we obtain $a_{-r} \times (a+r)_{m+r} = a_m$

$$\therefore \quad a_{-r} = \frac{a_m}{(a+r)_{m+r}}$$

If m and r be integers $a_{-r} = \frac{1}{(a+r)(a+r-1)\ldots(a+1)}$ which is analogous to $a^{-r} = \frac{1}{a \cdot a \cdot a \cdots}$ to r factors.

From the relations (α) and (β) we get

$$a_n \times (a-n)_m = a_m \times (a-m)_n$$

make $n = \frac{p}{q}$ then we have $\frac{a_{\frac{p}{q}}}{(a-m)_{\frac{p}{q}}} = \frac{a_m}{\left(a - \frac{p}{q}\right)_m}$

Supposing that m is a positive integer, this equation gives the ratio of any two functions $a_{\frac{p}{q}}$, $b_{\frac{p}{q}}$ in which a and b differ by an integer m, viz.:

$$a_{\frac{p}{q}} = b_{\frac{p}{q}} \cdot \frac{a \cdot a-1 \cdot a-2 \cdots a-m+1}{a-\frac{p}{q} \cdot a-\frac{p}{q}-1 \ldots a-\frac{p}{q}-m+1}$$

[redacted]

§ 5. Denote the infinite series (1) by $f(-n)$

thus $$f(-n) \equiv a_{-n} + (-n)_1 a_{-n-1} b_1 + \frac{(-n)_2}{2\,!} a_{-n-2} b_2 + \cdots \tag{A}$$

Now $$\left.\begin{aligned}(a+b+n)_m = (a+n+r)_m + m_1(a+n+r)_{m-1}(b-r)_1 \\ + \frac{m_2}{2\,!}(a+n+r)_{m-2}(b-r)_2 + \cdots\cdots + (b-r)_m\end{aligned}\right\} \tag{B}$$

m being a positive integer $> n$

Multiply $f(-n)$ by $(a+b+n)_m$ in the following manner:

a_{-n} by the series on the right side of B putting $r=0$

$(-n)_1 a_{-n-1} b_1$ ... ... $r=1$, etc.

Then we obtain

$$\left.\begin{aligned}&f(-n) \times (a+b+n)_m \\ &\equiv a_{-n}\left[(a+n)_m + m_1(a+n)_{m-1}b_1 + \frac{m_2}{2\,!}(a+n)_{m-2}b_2 + \ldots + b_m\right] \\ &+ (-n)_1 a_{-n-1} b_1 \\ &\quad \times \left[(a+n+1)_m + \quad \ldots \quad \ldots \quad \ldots \quad + (b-1)_m\right] \\ &+ \frac{(-n)_2}{2\,!} a_{-n-2} b_2 \\ &\quad \times \left[(a+n+2)_m + \quad \ldots \quad \ldots \quad \ldots \quad + (b-2)_m\right]\end{aligned}\right\} \tag{C}$$

+ an infinite number of brackets similar to the above.

Now with the interpretation of symbols in §§ (4) and (1).

$$a_{-n-s} \times (a+n+s)_m = a_{m-n-s} \quad \text{and} \quad b_s \times (b-s)_r = b_r$$

$\therefore$ The expression (C) becomes

$$\left[a_{m-n} + m_1 a_{m-n-1} b_1 + \frac{m_2}{2\,!} a_{m-n-2} b_2 + \ldots + a_{-n} b_m\right]$$

$$+ (-n)_1 \cdot \left[a_{m-n-1} b_1 + m_1 a_{m-n-2} b_2 + \quad \ldots \quad + a_{-n-1} b_{m+1}\right]$$

$$+ \frac{(-n)_2}{2\,!}\left[a_{m-n-2} b_2 + \quad \ldots \quad \ldots \quad \ldots \quad + a_{-n-2} b_{m+1}\right]$$

+ etc. to infinity.

Collect the resulting terms diagonally.

We obtain $\quad a_{m-n}$

$$+a_{m-n-1}b_1\Big[m_1+(-n)_1\Big]$$

$$+a_{m-n-2}b_2\left[\frac{m_2}{2\,!}+\frac{m_1(-n)_1}{1\,!\,1\,!}+\frac{(-n)_2}{2\,!}\right]$$

$$\cdots\cdots\cdots\cdots$$

$$+\quad a_0 b_{m-n}\left[\frac{m_{m-n}}{m-n\,!}+\cdots+\frac{(-r)_{m-n}}{m-n\,!}\right]$$

$+\cdots\cdot$ an infinite number of brackets similar to the above.

Now all brackets after the $m-n+1^{th}$ vanish identically by Vandermonde's Theorem since $m-n$ is a positive integer, and the expression becomes

$$a_{m-n}+(m-n)_1 a_{m-n-1}b_1+\frac{(m-n)_2}{2\,!}a_{m-n-2}b_2+\cdots+b_{m-n}\equiv(a+b)_{m-n}$$

$\therefore$ we have proved $\quad (a+b)_{m-n}=(a+b+n)_m\times f(-n)$

whence $\quad f(-n)=\dfrac{(a+b)_{m-n}}{(a+b+n)_m}=\dfrac{1}{(a+b+n)(a+b+n-1)\cdots(a+b+1)}$

which in our notation $=\ (a+b)_{-n}$

Conditions for the convergence of $f(-n)$ can easily be obtained.

we then obtain

$$x\left[1-{}_n\mathrm{C}_1+{}_n\mathrm{C}_2-\cdots\right]+\left[a-\frac{x}{x+1}{}_n\mathrm{C}_1(a-1)+\cdots\right]\equiv\frac{x\,!\,n\,!}{x+n\,!}(x+a)\,.$$

Now the first bracket on the left is identically equal to zero

$$\therefore\quad \frac{a}{x}-{}_n\mathrm{C}_1\frac{a-1}{x+1}+{}_n\mathrm{C}_2\frac{a-2}{x+2}-\cdots\text{ to } n+1 \text{ terms}\equiv\frac{x\,!\,n\,!}{x+n\,!}\,\frac{(x+a)}{x}$$

Proceeding in this way we shall finally obtain

$$\left[a^s-{}_n\mathrm{C}_1(a-1)^s+\cdots\cdots+(-1)^n(a-n)^s\right]$$
$$+\left[\frac{a^{s+1}}{x}-{}_n\mathrm{C}_1\frac{(a-1)^{s+1}}{x+1}+\cdots\cdots\right]\equiv\frac{x\,!\,n\,!}{x+n\,!}\;\frac{(x+a)^{s+1}}{x}$$

Now it is well known that the first bracket on the left $\equiv 0$ so long as s is an integer $<n$. $\therefore$ we have

$$\frac{a^p}{x}-{}_n\mathrm{C}_1\frac{(a-1)^p}{x+1}+{}_n\mathrm{C}_2\frac{(a-2)^p}{x+2}-\cdots\cdots\equiv\frac{x\,!\,n\,!}{x+n\,!}\;\frac{(x+a)^p}{x}\qquad\text{(D)}$$

p being an integer $\lesseqgtr n$.

This proves Theorem (9). Replacing $x+n$ by x and $a-n$ by a we obtain

$$\frac{a^p}{x}-{}_n\mathrm{C}_1\frac{(a+1)^p}{x-1}+\cdots\cdots\text{ to } n+1 \text{ terms}\equiv\frac{x-n\,!\,n\,!}{x\,!}\cdot\frac{(x+a)^p}{x-n}\qquad\text{(E)}$$

§ 7. Take $\mathrm{S}_1=a+b+c+\cdots$ to p terms

$\mathrm{S}_2=$ the sum of all products of the letters, two at a time.

. . . .

$\mathrm{S}_r=$ the sum of all products, r at a time.

Then

$$\mathrm{P}(x+n)=\mathrm{S}_p+(x+n)\mathrm{S}_{p-1}+(x+n)^2\mathrm{S}_{p-2}+\cdots+(x+n)^p\mathrm{S}_0\qquad\text{(F)}$$

From Theorem (9)

$$(x+n)^p \equiv x \cdot \frac{x+n\,!}{x\,!\,n\,!}\left\{\frac{n^p}{x} - n\mathrm{C}_1\frac{(n-1)^p}{x+1} + \cdots\cdots + (-1)^n\frac{0^p}{x+n}\right\}$$

so long as $p \mathrel{\overline{<}} n$. **Substitute in (F) then**

$$\mathrm{P}(x+n) \equiv \mathrm{S}_p\left[n^0 - \frac{x}{x+1}{}_n\mathrm{C}_1(n-1)^0 + \frac{x}{x+2}{}_n\mathrm{C}_2(n-2)^0 - \cdots\cdots\cdots\cdots + (-1)^n\frac{x}{x+n}0^0\right]\frac{x+n\,!}{x\,!\,n\,!}$$

$$+\,\mathrm{S}_{p-1}\left[n - \frac{x}{x+1}{}_n\mathrm{C}_1(n-1) + \cdots\cdots\cdots\cdots + (-1)^n\frac{x}{x+n}\cdot 0\right]\frac{x+n\,!}{x\,!\,n\,!}$$

$$+\,\mathrm{S}_{p-2}\left[n^2 - \frac{x}{x+1}{}_n\mathrm{C}_1(n-1)^2 + \cdots\cdots\cdots\cdots + (-1)^n\frac{x}{x+n}\cdot 0^2\right]\frac{x+n\,!}{x\,!\,n\,!}$$

$$\cdots\cdots\cdots\cdots$$

$$+\quad \mathrm{S}_0\left[n^p - \frac{x}{x+1}{}_n\mathrm{C}_1(n-1)^p + \cdots\cdots\cdots\cdots + (-1)^n\frac{x}{x+n}0^p\right]\frac{x+n\,!}{x\,!\,n\,!}$$

This expression may be written

If we replace a. by $\frac{a}{y}$, b by $\frac{b}{y}$, etc., then

$$P(r)=\frac{(a+ry)(b+ry)\cdots\cdot}{y^p}=\frac{P(ry)}{y^p}$$

and we obtain the theorem in form (2), since y^p is common to all denominators and so divides out.

Replacing $x+n$ by r, we obtain

$$\frac{P(0)}{r}-nC_1\frac{P(y)}{r-1}+nC_2\frac{P(2y)}{r-2}-\cdots\equiv(-1)^n\frac{r-n\,!\,n\,!}{r\,!}\;\frac{P(ry)}{r-n}\qquad(G)$$

§8. Again in Theorem (G) make

$$y=1,\quad r=x,\quad b=a-1,\quad c=a-2,\quad \text{etc.,}$$

then

$$P(ry)\equiv(a+r)(a+r-1)\cdots\cdot(a+r-p+1)\equiv(a+r)_p\equiv(a+x)_p$$

and we have

$$\frac{(a)_p}{x}-nC_1\frac{(a+1)_p}{x+1}+\text{ etc. }\cdots\cdot\equiv(-1)^n\frac{x-n\,!\,n\,!}{x\,!}\;\frac{(x+a)_p}{x-n}$$

This is the Identity (6).

If we replace $a+n$ by a, $b+n$ by $a-1$, $c+n$ by $a-2$, etc., we obtain

$$\frac{(a)_p}{x}-nC_1\frac{(a-1)_p}{x+1}+nC_2\frac{(a+2)_p}{x+2}-\cdots\cdot\equiv\frac{x\,!\,n\,!}{x+n\,!}\;\frac{(x+a)_p}{x}$$

This is the Identity (5).

The left side of (3) may be written

$$\begin{aligned}
&[S_p+0\cdot S_{p-1}+0^2S_{p-2}+\cdots\cdots\cdots+0^pS_0]\\
&-nC_1[S_p+yS_{p-1}+y^2S_{p-2}+\cdots\cdots\cdots\cdots+y_pS_0]\\
&+(-1)^n nC_n[S_p+nyS_{p-1}+n^2y^2S_{p-2}+\cdots+n^py^pS_0]\\
\equiv\ &S_p[1-nC_1+nC_2-\cdots]+S_{p-1}[0-nC_1\cdot y+nC_2\,.\,2y-\cdots]+\cdots\cdots\\
&\cdots\cdots+S_0[0^p-nC_1y^p+nC_2 2^py^p-\cdots]
\end{aligned}$$

when p is an integer $<n$ each bracket $\equiv 0$

$$\therefore\ P(0)-nC_1P(y)+nC_2P(2y)-\cdots\equiv 0.\qquad\text{Theorem (3)}$$

If $p=n$ all the brackets vanish except the last, which becomes

$$S_z[0^n - nC_1y^n + nC_2 2^n y^n - \cdots + (-1)^n n^n y^n] = (-y)^n n! \quad \text{Theorem (4)}$$

making $a=x$, $b=x-1$, $c=x-2$, etc., we have

$$(x)_n - nC_1(x+y)_n + nC_2(x+2y)_n - \cdots \equiv (-y)^n n!$$
$$(x)_p - nC_1(x+y)_p + \cdots\cdots \qquad \equiv 0. \quad \text{Theorems (7) and (8)}$$

Many other theorems can be obtained by varying the constants in (2) and (3).

§ 9. The Differential Equation of the n^{th} order

$$\left.\begin{aligned}(q_0 - p_0x)x^{n-1}\frac{d^ny}{dx^n} + (q_1 - p_1x)x^{n-2}\frac{d^{n-1}y}{dx^{n-1}} + \cdots\cdots \\ \cdots\cdots + (q_{n-1} - p_{n-1}x)\frac{dy}{dx} - p_ny = 0\end{aligned}\right\} \quad \text{(E)}$$

affords another very interesting analogy between powers, and products of related quantities.

When the constants p and q have the following values

[redacted]

When the constants in the differential equation have the following values

$$p_n = (a)_n \cdot \qquad\qquad q_{n-1} = (\gamma)_{n-1}$$

$$p_{n-1} = (a+1)_n - (a)_n \cdot \qquad\qquad q_{n-2} = (\gamma+1)_{n-1} - (\gamma)_{n-1}$$

$$p_{n-2} = \frac{(a+2)_n}{2\,!} - \frac{(a+1)_n}{1\,!\,1\,!} + \frac{(a)_n}{2\,!} \cdot \qquad\qquad \cdots\cdots\cdots\cdots$$

Etc. Etc.

The equation (E) has a solution

$$y = \mathrm{A}\left\{1 + \frac{(a)_n}{1\,!\,(\gamma)_{n-1}}x + \frac{(a)_n \cdot (a+1)_n}{2\,!\,(\gamma)_{n-1}(\gamma+1)_{n-1}}x^2 + \cdots\right\} \quad \cdots \quad \text{(G)}$$

The series (F) and (G) are particular cases of the Hypergeometric series of the n^{th} order.

On the Conditions that a given Straight Line may be a Normal to the Quadric Surface

$$(a, b, c, d, f, g, h, u, v, w)(x, y, z, 1)^2 = 0.$$

By R. H. Pinkerton, M.A.

Let the straight line be defined by the coordinates (α, β, γ) of a point on it and by its direction cosines l, m, n. It may be referred to as the line $(\alpha, \beta, \gamma, l, m, n)$. Write, for shortness, the equation to the quadric surface in the form $F(x, y, z) = 0$.

The line $(\alpha, \beta, \gamma, l, m, n)$ will be a normal to the quadric if it is perpendicular to either of the tangent planes to the quadric at the points where it cuts the quadric. The equation to this pair of tangent planes may be found as follows:

The line $(\alpha, \beta, \gamma, l, m, n)$ will cut the quadric in two points (x', y', z'), whose distances, r, from (α, β, γ) are the roots of the equation $\qquad F(\alpha + lr, \beta + mr, \gamma + mr) = 0,$

that is, of the equation

$$r^2(lL + mM + nN) + 2r(lP + mQ + nR) + F(\alpha, \beta, \gamma,) = 0 \quad \ldots \quad (1)$$

in which

[redacted]

If we now eliminate r between the equations (1) and (2), we shall obtain the required equation to the pair of tangent planes in the form

$$\begin{aligned}(x\mathrm{P}+y\mathrm{Q}+z\mathrm{R}+\mathrm{S})^2(l\mathrm{L}+m\mathrm{M}+n\mathrm{N})\\ -2(x\mathrm{P}+y\mathrm{Q}+z\mathrm{R}+\mathrm{S})(x\mathrm{L}+y\mathrm{M}+z\mathrm{N}+\mathrm{K})(l\mathrm{P}+m\mathrm{Q}+n\mathrm{R})\\ +\mathrm{F}(\alpha,\beta,\gamma).(x\mathrm{L}+y\mathrm{M}+z\mathrm{N}+\mathrm{K})^2=0.\end{aligned}$$

On multiplying this equation by $l\mathrm{L}+m\mathrm{M}+n\mathrm{N}$, it will appear that it is equivalent to

$$\begin{aligned}&[(x\mathrm{P}+y\mathrm{Q}+z\mathrm{R}+\mathrm{S})(l\mathrm{L}+m\mathrm{M}+n\mathrm{N})\\ &\qquad -(x\mathrm{L}+y\mathrm{M}+z\mathrm{N}+\mathrm{K})(l\mathrm{P}+m\mathrm{Q}+n\mathrm{R})]^2\\ &=(x\mathrm{L}+y\mathrm{M}+z\mathrm{N}+\mathrm{K})^2[(l\mathrm{P}+m\mathrm{Q}+n\mathrm{R})^2\\ &\qquad -\mathrm{F}(\alpha,\beta,\gamma).(l\mathrm{L}+m\mathrm{M}+n\mathrm{N})],\end{aligned}$$

a form which clearly indicates two planes whose line of intersection is given by the equations

$$x\mathrm{P}+y\mathrm{Q}+z\mathrm{R}+\mathrm{S}=0$$
$$x\mathrm{L}+y\mathrm{M}+z\mathrm{N}+\mathrm{K}=0.$$

In the equation just found write

U for $l\mathrm{L}+m\mathrm{M}+n\mathrm{N}$,

V for $l\mathrm{P}+m\mathrm{Q}+n\mathrm{R}$,

and ρ^2 for $(l\mathrm{P}+m\mathrm{Q}+n\mathrm{R})^2-\mathrm{F}(\alpha,\beta,\gamma).(l\mathrm{L}+m\mathrm{M}+n\mathrm{N})$.

It follows from equation (1) that $\rho^2=0$ is the condition that the line $(\alpha,\beta,\gamma,l,m,n)$ may be a *tangent* line to the quadric, and that ρ is real or imaginary according as the line does or does not cut the quadric in real points.

On making the substitutions indicated we get as the equations to the tangent planes

$$\begin{aligned}(xP+y\mathrm{Q}+z\mathrm{R}+\mathrm{S})\mathrm{U}-(x\mathrm{L}+y\mathrm{M}+z\mathrm{N}+\mathrm{K})\mathrm{V}\\ =\pm(x\mathrm{L}+y\mathrm{M}+z\mathrm{N}+\mathrm{K})\rho,\end{aligned}$$

or
$$\begin{aligned}x(\mathrm{PU}-\mathrm{LV}\pm\mathrm{L}\rho)+y(\mathrm{QU}-\mathrm{MV}\pm\mathrm{M}\rho)\\ +z(\mathrm{RU}-\mathrm{NV}\pm\mathrm{N}\rho)+(\mathrm{SU}-\mathrm{KV}\pm\mathrm{K}\rho)=0,\end{aligned}$$

in which the signs in the ambiguities are to be taken all $+$ or all $-$.

Now the line $(\alpha, \beta, \gamma, l, m, n)$ is a normal to the quadric if it is perpendicular to either of these planes. Hence the line will be a normal if *one* of the two following sets of conditions is fulfilled:

$$(\mathrm{PU} - \mathrm{LV} + \mathrm{L}\rho)/l = (\mathrm{QU} - \mathrm{MV} + \mathrm{M}\rho)/m = (\mathrm{RU} - \mathrm{NV} + \mathrm{N}\rho)/n,$$

$$(\mathrm{PU} - \mathrm{LV} - \mathrm{L}\rho)/l = (\mathrm{QU} - \mathrm{MV} - \mathrm{M}\rho)/m = (\mathrm{RU} - \mathrm{NV} - \mathrm{N}\rho)/n.$$

It may be noticed that if *both* sets of conditions are fulfilled, the line is an axis of the quadric, for the line $(\alpha, \beta, \gamma, l, m, n)$ is then perpendicular to both of the tangent planes at the points where it meets the surface. Hence we arrive at the known conditions that the line $(\alpha, \beta, \gamma, l, m, n)$ may be an axis, viz.,

$$\mathrm{L}/l = \mathrm{M}/m = \mathrm{N}/n,$$

and

$$\mathrm{P}/l = \mathrm{Q}/m = \mathrm{R}/n.$$

Additional Note on Triangle Transformations.

By R. F. Muirhead, M.A.

Part I.

The chief object of this Note is to develope some simple and rather interesting properties of the operators α, β, γ, referred to in my Note read at last meeting. I take for brevity the symbol μ to denote the compound operator

$$\alpha\beta\gamma \text{ or its equivalents } \gamma\alpha\beta, \ \beta\alpha\gamma, \text{ etc.}$$

1. If we apply μ to any function F, of $a, b, c, s, s_1, s_2, s_3, r, r_1, r_2, r_3, h_1, h_2, h_3, \Delta$, R; then, as I pointed out before, $\mu F = F$, if F be an *even* function of certain letters; but if F be an *odd* function, then $\mu F = -F$; and lastly, if $F \equiv F_1 + F_2$, where F_1 is odd and F_2 even, then $\mu F = -F_1 + F_2$.

2. The order of the operations α, β, γ successively applied, is immaterial.

If F be any function of the letters above mentioned, we have

$$\alpha^2 F = F, \text{ or } \alpha^2 = 1 \ \therefore \alpha = \frac{1}{\alpha}.$$

Again $\quad \beta\gamma\gamma\beta = \beta\gamma^2\beta = \beta\beta = 1 \quad \therefore \alpha\beta\gamma\gamma\beta = \alpha$

$$\therefore \mu\gamma\beta = \alpha.$$

Similarly $\quad \mu\beta\gamma = \alpha \quad \therefore \gamma\beta = \beta\gamma.$

Thus any two successive operators may be transposed without altering the result. Hence the order of any number of successive operations is immaterial.

3. Next, any succession of operations is reducible to one of eight operations. For by the last paragraph, any such succession is equivalent to $\alpha^m\beta^n\gamma^p$, where m, n, p are positive integers.

But $\alpha^m = 1$ or α according as m is even or odd. Hence $\alpha^m\beta^n\gamma^p$ reduces to one of the following eight operations:

$$1,\ \alpha,\ \beta,\ \gamma,\ \beta\gamma,\ \gamma\alpha,\ \alpha\beta,\ \alpha\beta\gamma;$$

which may be written

$$1,\ \alpha,\ \beta,\ \gamma,\ \mu\alpha,\ \mu\beta,\ \mu\gamma,\ \mu.$$

4. These form a "group" of operations, *i.e.*, any combination of them is equivalent to one or other of the eight.

And using the nomenclature explained by F. Klein in his "Lectures on the Icosahedron;" the *periodicity* of each operation (excepting the first) is 2; the most extended "*sub-groups*" are of the type $1,\ \alpha,\ \mu\alpha,\ \mu$; which again contains two *sub-groups* $1,\ \alpha$ and $1,\ \mu$.

5 When the operation is performed on an *even* function, then $\mu = 1$, and the group reduces to $1,\ \alpha,\ \beta,\ \gamma$; with $1,\ \alpha$, etc., as sub-groups: and if on an *odd* function, then $\mu = -1$, and the group is $\pm 1,\ \pm\alpha,\ \pm\beta,\ \pm\gamma$.

6. As pointed out at the end of the preceding Note, the angles ABC may occur in any functions which unaltered in absolute magnitude when $-\mathrm{A}$ is changed into $-\mathrm{A} + 2\pi$, etc., without modifying the above results; noting that A, B, C are reckoned along with r_1, r_1, etc., in counting the dimensions as *even* or *odd*.

[redacted]

This, though similar to the transformation α in a plane triangle, is not strictly analogous.

8. Referring to the demonstration of the principle of the *transformation continue* as given by Mr Lemoine in the Report of the French Association for the Advancement of Science for 1891, Vol. II., p. 118, and ascribed by him to M. Laisant, it was pointed out that a somewhat simpler point of view is possible: from which it appears that many other kinds of transformations are equally valid. As interesting cases, the following were mentioned:

(1) ABC changed to $\frac{\pi}{2}-\frac{A}{2}$, $\frac{\pi}{2}-\frac{B}{2}$, $\frac{\pi}{2}-\frac{C}{2}$ and a into $a\operatorname{cosec}\frac{A}{2}$, etc, which corresponds to changing from the given triangle to that whose vertices are the excentres of ABC.

(2) s_1, s_2, s_3 changed to r_1, r_2, r_3.

Some properties of both were mentioned.

9. The suggestion made at the close of the preceding Note was verified by the example

$$\sin\frac{A}{2}=\sqrt{\frac{s_2 s_3}{bc}}$$

in which, after the transformation α, the square root must be taken *negatively*.

Added Note.—Dr Mackay having pointed out that α, β, γ have already been appropriated to denote certain quantities connected with the triangle, the author suggests the symbols t_1, t_2, t_3 to replace α, β and γ, in cases where these symbols have other meanings already ascribed to them.

Examples of a Method of Developing Logarithms and the Trigonometrical Functions without the Calculus by means of their Addition Formulae and Indeterminate Coefficients.

By John Jack, M.A.

[Abstract.]

The convergence of the series is *assumed*.

The method consists in assuming that the function is equal to a certain power series with undetermined coefficients, substituting these series in the addition formula. This gives an identity.

Ex. gr. $\sin^{-1}x + \sin^{-1}y = \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$

$$\therefore \quad \left.\begin{array}{l} a_1x + a_2x^2 + a_3x^3 + \cdots \\ + a_1y + a_2y^2 + a_3y^3 + \cdots \end{array}\right\}$$

$$= \Big\{ a_1(x\sqrt{1-y^2} + y\sqrt{1-x^2}) + a_2(x\sqrt{1-y^2} + y\sqrt{1-x^2})^2$$

[redacted]

done of course in practically the same way, but is given on account of its intimate connection with No. 1. I give

(1) $\log x$	(3) $\sin^{-1}x$	(5) $\tan^{-1}x$
(2) $\log^{-1}x$ or e^x	(4) $\sin x$, $\cos x$	(6) $\tan x$.

The method seems symmetrical and quite elementary. The analogy between sin, cos, tan, and sinh, cosh, tanh, can be readily seen without at all using the imaginary i, by developing by this plan.

1. To develop $\log\overline{1+x}$ in a series of powers of x

$$\log(1+x)(1+y) = \log(1+x) + \log(1+y)$$

Let $$\log(1+x) = \phi(x)$$

$\therefore$ $$\phi(x) + \phi(y) = \phi(x + y + xy)$$
$$= \phi(x + y\overline{1+x})$$

Let $$\phi(x) = a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots$$

$\therefore$ $$\left.\begin{matrix} a_1x + a_2x^2 + a_3x^3 + \cdots \\ + a_1y + a_2y^2 + a_3y^3 + \cdots \end{matrix}\right\} \equiv \left\{\begin{matrix} a_1(x + y\overline{1+x}) + a_2(x + y\overline{1+x})^2 \\ + a_3(x + y\overline{1+x})^3 + \cdots \end{matrix}\right.$$

Pick out the coefficient of y.

$\therefore$ $$a_1 = (1+x)(a_1 + 2a_2x + 3a_3x^2 + \cdots)$$
$$= a_1 + (a_1 + 2a_2)x + (2a_2 + 3a_3)x^2 + \cdots$$

and the coefficients of x must vanish

$\therefore$ $$a_1 + 2a_2 = 0$$
$$2a_2 + 3a_3 = 0$$
$$3a_3 + 4a_4 = 0$$
$$4a_4 + 5a_5 = 0$$

and so on.

$\therefore$ $$a_1 = -2a_2 = 3a_3 = -4a_4 = 5a_5 = -,\cdots$$

$\therefore$ $$a^2 = -\frac{a_1}{2},\ a_3 = \frac{a_1}{3},\ a_4 = -\frac{a_1}{4},\ a_5 = \frac{a_1}{5},\ \text{and so on}$$

$$\therefore \qquad \phi(x) = a_1\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} \cdots\right)$$

$$\therefore \qquad \log(1+x) = a_1\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} \cdots\right)$$

and a_1 must be determined otherwise.

The other expansions are given in abstract.

2. To find the number corresponding to a logarithm, or to develop $log^{-1}x$.

Taking as before $\qquad \phi(x) = \log(1+x)$

$$\phi(x) + \phi(y) = \phi(x + y + xy)$$

Let $\qquad \phi(x) = u \qquad \therefore \quad x = \phi^{-1}(u)$

$\qquad \phi(y) = v \qquad \therefore \quad y = \phi^{-1}(v)$

$$\therefore \qquad u + v = \phi\{\phi^{-1}(u) + \phi^{-1}(v) + \phi^{-1}(u)\phi^{-1}(v)\}$$

$$\therefore \qquad \phi^{-1}(u+v) = \phi^{-1}(u) + \phi^{-1}(v) + \phi^{-1}(u) \cdot \phi^{-1}(v)$$

Let $\qquad \phi^{-1}(\ ) = a_1(\ ) + a_2(\ )^2 + a_3(\ )^3 + \cdots$

Insert these expansions in the equation just given, and pick out the coefficients of v.

From the identity so obtained in powers of u, we get, by

[redacted]

3. Required the development of $\sin^{-1}x$. By similar treatment of the identity

$$\sin^{-1}x + \sin^{-1}y = \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$$

we get

$$\sin^{-1}x = a_1\left(x + \frac{1}{2}\frac{x^3}{3} + \frac{3}{2^3}\frac{x^5}{5} + \frac{5}{2^4}\frac{x^7}{7} + \frac{35}{2^7}\frac{x^9}{9} + \frac{63x^{11}}{2^8 11} + \frac{231x^{13}}{2^{10}13} + \cdots\right)$$

$$= a_1\left(x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \cdots\right)$$

and a_1 is otherwise found to be 1.

$\sin^{-1}x$ is thus found to be an odd function of x.

4. The development of $\sin u$, $\cos u$. This is got from the identity

$$\sin^{-1}x + \sin^{-1}y = \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$$

or

$$u + v = \sin^{-1}(\sin u \cos v + \cos u \sin v.$$

It is shown, first, that 'sin$u$ is an odd function of u, and $\cos u$ an even function of u. The series are then assumed, and the coefficients evaluated as above.

5. The development of $\tan^{-1}x$. This is got from the identity

$$\tan^{-1} + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$$

It is first established that $\tan^{-1}x$ is an odd function of x, and then the series is assumed and the coefficients evaluated in the usual way.

6. The development of $\tan u$. Here we have

$$\tan u + \tan v = \tan(u+v) - \tan u \tan v \tan(u+v).$$

Assume the series involving odd powers, and proceed as above.

The paper ended with an expansion in terms of arcs of small tangents, for calculating π.

Fifth Meeting, March 8th, 1895.

JOHN M'COWAN, Esq., M.A., D.Sc., President, in the Chair.

Some Formulae in connection with the Parabolic Section of the Canonical Quadric.

By CHARLES TWEEDIE, M.A., B.Sc.

§ 1. I have ventured to bring the formulae of this paper before the Society, as I have been unable to find reference to them in any text-book or any original contribution to mathematical literature which I have come across. I confine my attention completely to the central surface, as the corresponding formulae for the paraboloids are very readily deduced by a similar process.

§ 2. Let the equation to the quadric be

[redacted]

Then S can not vanish for real values of $l\,m\,n$, and the following relations may be established:

$$S = \Sigma a^2 l^2 = \Sigma m^2 n^2 (\beta - \gamma)^2 = \sqrt{-\Sigma m^2 n^2 \beta\gamma(\beta - \gamma)^2} \quad \ldots \quad \ldots \quad (A)$$

$$= \beta m^2(\beta - a) + \gamma n^2(\gamma - a) = \text{etc.} = n^2(\gamma - a)(\gamma - \beta) - a\beta = \text{etc.}$$

$$\beta m^2(\beta - a)^2 + \gamma n^2(\gamma - a)^2 = S(\gamma + \beta - a) + a\beta\gamma, \text{ etc.} \quad \ldots \quad \ldots \quad (B)$$

These might be added to, but they are all that are made use of in what follows.

Vertex.

§ 3. The coordinates of the vertex may be found by finding the equation of the plane in which the vertices of the parallel parabolic sections lie (just as for elliptic or hyperbolic sections), and solving for x, y, z with the aid of (1) and (2).

Now the axis of the parabola is parallel to what would be the diameter of the section as on the quadric, and therefore its direction cosines are $(\rho a l, \rho\beta m, \rho\gamma n)$, where ρ^2 is therefore equal to $1/S$.

If (x, y, z) be the vertex, the tangent to the parabola is perpendicular to $(\rho a l)$

i.e., $(\rho a l)$ is perpendicular to the direction given by the intersection of

$$lx + my + nz = p$$

and
$$\frac{xx_1}{a} + \frac{yy_1}{\beta} + \frac{zz_1}{\gamma} = 1$$

and therefore the vertex satisfies the equation

$$\Sigma \frac{x}{al}(\beta - \gamma) = 0 \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

Solving (2) and (3) for y and z in terms of x, and remembering (A), we deduce

$$y/\beta m = x/al - p(a - \beta)/S \quad \ldots \quad \ldots \quad (4)$$

$$z/\gamma n = x/al - p(a - \gamma)/S \quad \ldots \quad \ldots \quad (5)$$

Substitute these values in (1); we have a quadratic in x, which, however, must have one infinite root, and which, in virtue of (II) and (A) reduces to

$$2px/\alpha l = 1 - p^2\{\beta m^2(\alpha-\beta)^2 + \gamma n^2(\alpha-\gamma)\}^2/S^2$$

$$= 1 - \frac{p^2}{S}(\gamma+\beta-\alpha) - \alpha\beta\gamma p^2/S^2 \qquad \text{by (B).} \qquad \ldots \qquad (6)$$

Similarly

$$\frac{2py}{\beta m} = 1 - p^2(\alpha+\gamma-\beta)/S - \alpha\beta\gamma p^2/S^2. \qquad \ldots \qquad (7)$$

$$\frac{2pz}{\gamma n} = 1 - p^2(\alpha+\beta-\gamma)/S - \alpha\beta\gamma p^2/S^2. \qquad \ldots \qquad (8)$$

These are the coordinates of the vertex.

Parameter.

§ 4. Let V be the vertex, F the focus, FL the semi-latus rectum.

If (λ) be the direction of the tangent at the vertex, it will also be that of FL

Now (λ) is perpendicular to (l) of plane and to (αl) of [redacted]

But L is on the quadric, and V is on the quadric

$$\therefore \quad \Sigma\frac{1}{a}\{x+\pi\rho al+2\pi\rho mn(\beta-\gamma)\}^2=1$$

$$\therefore \quad 2\pi\rho\left\{\Sigma lx+4\Sigma\frac{mnx}{a}(\beta\quad\gamma)\right\}$$
$$+\pi^2\rho^2\left\{\Sigma al^2+4lmn\Sigma(\beta-\gamma)+\Sigma\frac{m^2n^2(\beta-\gamma)^2}{a}\right\}=0$$

which by (2) and (3) reduces to

$$2\pi\rho p+\frac{4\pi^2\rho^2}{a\beta\gamma}\Sigma m^2n^2\beta\gamma(\beta-\gamma)^2=0$$

and $\therefore$ $$4\pi=-2a\beta\gamma p/(-S^2)\rho$$
$$=\ 2a\beta\gamma p\rho^3 \quad \text{by (A).} \quad \ldots \quad (11)$$

So that for a system of parallel parabolic sections the parameter is not constant, as I had at first supposed, but varies directly as the distance of the section from the origin.

For the central section itself $p=0$, *i.e.* the parameter vanishes. The section then reduces to a pair of parallel lines for the hyperboloid of one sheet, which are coincident asymptotes in the two-sheeted surface. In the former there are, therefore, an infinite number of pairs of parallel generators, and, from the present analysis, one might infer that the parameter of a pair of parallel lines, considered as the limiting case of a parabola, is zero.

Focus.

§ 5. The coordinates of the focus are given by

$$\xi=x+\pi\rho al, \text{ etc.}$$

They are therefore given by

$$\left.\begin{aligned}\frac{2p\xi}{al}&=1-\frac{p^2}{S}(\gamma+\beta-a)\\ \frac{2p\eta}{\beta m}&=1-\frac{p^2}{S}(\gamma+a-\beta)\\ \frac{2p\zeta}{\gamma n}&=1-\frac{p^2}{S}(a+\beta-\gamma).\end{aligned}\right\} \quad (12)$$

If we consider l, m, and n as connected by the equations (I) and (II) these contain implicitly equations (2) and (3).

For a system of parallel sections the locus of the foci is a plane curve, and the orthogonal projection on a reference plane as obtained by eliminating p between any two of the equations (12) is found to be a hyperbola whose centre is at the origin. Hence the locus of the foci of a series of parallel parabolic sections $\Sigma lx = p$ is a hyperbola whose centre is at the origin, and which lies in the plane

$$\Sigma\frac{x}{al}(\beta-\gamma)=0.$$

Moreover $\Sigma lx = 0$ and $\Sigma\frac{x}{al}(\beta-\gamma)=0$ are conjugate planes, and hence the foci for any system of parallel plane sections parallel to $\Sigma lx = 0$ is a conic lying in a plane conjugate to $\Sigma lx = 0$.

§ 6. The equation $\Sigma\frac{x}{al}(\beta-\gamma)=0$ might be considered as given along with the equation $\Sigma al^2 = 0$; then we know that in it lies a hyperbola which is the locus of the foci of a system of parabolic sections. To find such a system, all we have to do is to take the diameter conjugate to the given plane, and draw the planes passing through it and touching the asymptotic cone. There are two such planes, real or coincident, since one is $\Sigma lx = 0$. They are not coincident since that would require that the diameter in question $\left(\frac{\beta}{l}, \frac{\gamma}{m}\right)$ should be a generator of the cone and $\therefore$ $\Sigma m^2n^2\beta\gamma(\beta-\gamma)^2 = 0$,

[redacted]

any five independent equations involving them. This surface would seem to be of a fairly high degree, but, whatever its degree, every plane section whose plane is at the same time tangent to the cone (13) must be a degenerate curve, part or whole of whose real intersection consists of two concentric hyperbolas, with centre at the origin.

Surface of Revolution.

§ 8. For the surface of revolution, the formulae are much simpler. Suppose that $\beta = \alpha$, then $1/\rho^2 = S = -\alpha\gamma$, and $\therefore$ S is constant.

Equations (I) and (II) become

$$l^2 + m^2 + n^2 = 1 \quad \ldots \quad \ldots \quad \ldots \quad (I)'$$

$$\alpha(l^2 + m^2) + \gamma n^2 = 0 \quad \ldots \quad \ldots \quad \ldots \quad (II)'$$

whence

$$\left.\begin{aligned} n^2 &= -\alpha/(\gamma - \alpha) \\ l^2 + m^2 &= \gamma/(\gamma - \alpha) \end{aligned}\right\} \quad \ldots \quad (14)$$

so that n^2 is constant, as is otherwise obvious.

The parameter

$$4\pi = 2\alpha\beta\gamma p\rho^3 \qquad \text{becomes}$$

$$4\pi = 2\alpha\beta\gamma p \frac{1}{\sqrt{-\alpha^3\gamma^3}} = \frac{2\alpha p}{\sqrt{-\alpha\gamma}} = 2p\sqrt{-\frac{\alpha}{\gamma}}, \quad \ldots \quad (15)$$

and therefore varies directly as the distance of the section from the origin.

The coördinates of the focus are given by

$$\frac{2p\xi}{\alpha l} = 1 - \frac{p^2}{-\alpha\gamma}\gamma = 1 + \frac{p^2}{\alpha} \quad \ldots \quad \ldots \quad (16)$$

$$\frac{2p\eta}{\alpha m} = \quad \ldots \quad = 1 + \frac{p^2}{\alpha} \quad \ldots \quad \ldots \quad (17)$$

$$\frac{2p\zeta}{\gamma n} = \quad 1 + \frac{p^2}{\alpha\gamma}(2\alpha - \gamma) \quad \ldots \quad \ldots \quad (18)$$

so that, (n being constant), when p is constant so also is ζ.

§ 9. Equations (16) and (17) give $l\eta - m\xi = 0$, and hence the cone of equation (13) shrinks up into the z-axis for the surface of revolution.

From the same equations we deduce

$$\frac{\xi}{l} = \frac{n}{m} = \sqrt{\frac{\xi^2+\eta^2}{l^2+m^2}} = cr, \text{ say, where}$$

$$r^2 = \xi^2 + \eta \quad \text{and} \quad c^2 = 1/(l^2+m^2) = (\gamma - \alpha)/\gamma.$$

Hence, substituting in (16) and (18), we deduce

$$p^2 - 2pcr + \alpha = 0 \qquad \dots \qquad \dots \qquad (19)$$

$$p^2(2\alpha - \gamma) - 2p\frac{\alpha\zeta}{n} + \gamma\alpha = 0 \qquad \dots \qquad \dots \qquad (20)$$

On eliminating p from these two equations, we obtain the locus of the foci of all the parabolic sections.

The eliminant in question is

$$\left(\gamma\alpha - \alpha\cdot\overline{2\alpha-\gamma}\right)^2 = \left(-2cr\alpha\gamma + \frac{2\alpha^2\zeta}{n}\right)\left(-2\frac{\alpha\zeta}{n} + 2cr\cdot\overline{2\alpha-\gamma}\right) \qquad (21)$$

and this, on replacing c and n by their equivalents in terms of α and γ, reduces to

$$\alpha(\gamma-\alpha) - \alpha\zeta^2 - r^2\gamma + 2\alpha r^2 = 2\alpha^2\zeta r/(1-\alpha\gamma). \qquad \dots \qquad (22)$$

[redacted]

Some Suggestions in Mathematical Terminology.

By R. F. Muirhead, M.A.

[Abstract.]

1. To designate the line which bisects at right angles the join of two points A, B, the term *axis of A, B* is proposed. Reasons :—(1) brevity; (2) avoidance of the suggestion that the line joining AB is necessary in constructing it (important in teaching Geometrical Drawing); (3) two points, like any other pair of circles, have a *radical axis* which is the line in question.

(2) To designate the tangent of the angle of inclination to the horizontal, or to the x-axis, or to any straight line or direction of reference: *gradient* is proposed. Reasons :—(1) Present Engineering usage; (2) as compared with the word *slope* it has a more definite suggestion of the way it is to be measured, *i.e.*, by trigonometrical tangent, not by angular magnitude; (3) already in use by several authorities.

(3) For unit of *moment of a force* on the British system of units: *pound-foot*, and *poundal-foot* for the gravitational and absolute unit respectively.

(4) For inverse trigonometrical and hyperbolic functions, *nis*, *soc*, *nat*, *toc*, *ces*, *cesoc*, *nish*, *soch*, *nath*, *toch*, *cesh*, *cesoch*, in place of sin^{-1}, cos^{-1}, etc., and of the continental *arc sin*, *arc sin hyp*, etc.

A Suggestion for the Improvement of Mathematical Tables.

By W. J. Macdonald, M.A.

If in Tables such as Chambers' the differences were given for a second instead of for a minute, they might be arranged (and used) as follows:

30°	Lsin	Diff.
0′	9 6989700	1″ 36
1′	9·6991887	2″ 72
2′	9 6994073	3″ 108
3′	9·6996258	4″ 144
4′	9·6998441	5″ 180
5′	9·7000622	6″ 216
6′	9·7002802	7″ 252

[redacted]

To find the angle whose

$$\begin{aligned} \text{Lsin} &= 9{\cdot}7005632 \\ \text{Lsin}30°7' &= 9{\cdot}7004981 \\ \hline & \phantom{9{\cdot}7004}651 \\ 10'' = & \phantom{9{\cdot}7004}360 \\ \hline & \phantom{9{\cdot}7004}291 \\ 8'' = & \phantom{9{\cdot}7004}288 \end{aligned}$$

$\therefore$ the angle is $30°7'18''$.

The following Resolution, moved by Mr DUTHIE, was unanimously adopted :—

"Considering that in the tables of lineal and square measure the fractional measures—viz., $5\frac{1}{2}$ yds. = 1 pl., and $30\frac{1}{4}$ sq. yds. = 1 sq. pl.—are of no practical value, and in their premature appearance in arithmetical study involve a grievous and unnecessary burden in teaching, this Society appeal to the Scottish Education Department to exercise its authority with a view to their abolition in schools, and to this end to allow no questions involving their use to be set in examinations under the control of the Department."

Sixth Meeting, April 10th, 1895.

JOHN M'COWAN, Esq., M.A., D.Sc., President, in the Chair.

On the Operation of Division.

By JOHN M'COWAN, M.A., D.Sc.

Sur les cubiques gauches équilatères.

By CH. BIOCHE.

J'appelle *cubiques gauches équilatères* les cubiques qui ont trois asymptotes rectangulaires deux à deux. Ces cubiques gauches possèdent des propriétés qui rappellent les propriétés classiques de

[redacted]

par deux points arbitrairement choisis $(X_1Y_1Z_1)$ $(X_2Y_2Z_2)$. On a alors trois groupes de deux équations telles que

$$A + aX_1 = t_1X_1$$
$$A + aX_2 = t_2X_2 .$$

Les deux équations que je viens d'écrire donnent A et a, si X_1 est différent de X_2. D'ailleurs on peut remarquer que si on avait $X_1 = X_2$, il y aurait une droite rencontrant la courbe en trois points, cette courbe ne serait donc pas une cubique gauche.

2. Supposons que l'on considère quatre points 1, 2, 3, 4 de la courbe E, correspondant à des valeurs t_1, t_2, t_3, t_4 du paramètre variable. Les coefficients directeurs de la corde (1, 2) sont proportionals à

$$\frac{A}{(t_1-\alpha)(t_2-\alpha)}, \quad \frac{B}{(t_1-\beta)(t_2-\beta)}, \quad \frac{C}{(t_1-\gamma)(t_2-\gamma)}$$

Donc la condition pour que deux cordes (1, 2), (3, 4) soit rectangulaires s'exprime par l'equation

$$\text{(I)} \quad \frac{A^2}{(t_1-\alpha)(t_2-\alpha)(t_3-\alpha)(t_4-\alpha)} + \frac{B^2}{(t_1-\beta)(t_2-\beta)(t_3-\beta)(t_4-\beta)} + \frac{C^2}{(t_1-\gamma)(t_2-\gamma)(t_3-\gamma)(t_4-\gamma)} = 0$$

L'interprétation de cette équation donne diverses propriétés géométriques.

3. D'abord remarquons qu'elle est symétrique par rapport aux quatre indices. D'ici il resulte que les six arêtes du tétraèdre (1, 2, 3, 4) sont orthogonales.

Donc, *si deux cordes d'une cubique équilatère sont orthogonales, les extrémités de ces cordes sont les sommets d'un tétraèdre à arêtes opposées orthogonales ;* ou autrement dit *les droites qui joignent les extrémités de ces cordes sont deux à deux orthogonales.* Sous cette dernière forme l'enoncé s'applique, sans modification, au cas de l'hyperbole équilatère.

4. L'équation(I) est du 2ᵉ degré par rapport à t_4, par example; autrement dit, si on se donne trois points 1, 2, 3, il y a deux points M et M′ qui peuvent former avec les trois premiers un tétraèdre à arêtes opposées orthogonales. Or on sait que chaque sommet d'un tel tétraèdre se projette au point de rencontre des hauteurs de la face opposée. Donc M et M′ sont sur la perpendiculaire élevée sur le plan 1, 2, 3, au point de rencontre des hauteurs du triangle correspondant. Donc *la corde d'une cubique équilatère qui est perpendiculaire à un plan, rencontre ce plan au point de concours des hauteurs du triangle formé par ses trois points d'intersection avec la cubique.*

Inversement, *le lieu des points de concours des hauteurs des triangles déterminés par la cubique sur des plans parallèles est la corde perpendiculaire à ces plans.*

En particulier on voit qu'il y a deux plans, parmi ceux qui sont parallèles à un plan donné qui coupent une cubique équilatère en trois points formant un triangle rectangle. Ce sont ceux qui passent par les extrémités de la corde correspondante. Si cette corde devient tangente il n'y a plus qu'un plan, c'est le plan normal.

Le théorème général qui précède peut s'énoncer encore de la façon suivante, *si l'on projette orthogonalement une cubique équilatère sur un plan le point double de la projection est le point de concours des hauteurs du triangle formé par les points où le plan de projection coupe la cubique.*

[redacted]

Il suffit de démontrer le théorème pour le cas où le sommet de l'angle est à l'origine puisque l'origine est un point quelconque de la courbe.

Soient $(X_1Y_1Z_1)$, $(X_2Y_2Z_2)$, $(X_3Y_3Z_3)$ les points où les arêtes du trièdre rencontrent la cubique. Ces arêtes étant deux à deux rectangulaires, on a trois relations de la forme

$$X_1X_2 + Y_1Y_2 + Z_1Z_2 = 0$$

ou $$\frac{A^2}{(t_1-\alpha)(t_2-\alpha)} + \frac{B^2}{(t_1-\beta)(t_2-\beta)} + \frac{C^2}{(t_1-\gamma)(t_2-\gamma)} = 0$$

Or si l'on remarque que

$$\frac{A}{t_1-\alpha} - \frac{A}{t_2-\alpha} = \frac{(t_2-t_1)A}{(t_1-\alpha)(t_2-\alpha)}$$

on voit que l'équation précédente peut s'écrire

$$A \cdot \frac{A}{t_1-\alpha} + B \cdot \frac{B}{t_1-\beta} + C \cdot \frac{C}{t_1-\gamma} = A \cdot \frac{A}{t_2-\alpha} + B \cdot \frac{B}{t_2+\beta} + C \cdot \frac{C}{t_2-\gamma}$$

ou $$AX_1 + BY_1 + CZ_1 = AX_2 + BY_2 + CZ_2 .$$

Ces expressions sont évidemment égales à

$$AX_3 + BY_3 + CZ_3 .$$

Les trois points considerés sont donc à la même distance du plan

$$AX + BY + CZ = 0$$

or il est facile de vérifier que ce plan est le plan normal.

Isoperimetric $2^m n$-gons applied to finding $\dfrac{1}{\pi}$ concisely by a new construction.

By R. E. Anderson, M.A.

Figure 27.

1. Let AB be the half-side of any n-gon, OB its in-radius (r), and OA its circum-radius (R). Draw OA_1 to bisect $\angle AOB$ and $AA_1C \perp$ to it meeting OB in C. Then $A_1B_1 \parallel$ to AB is the half-side of a $2n$-gon having the same perimeter as the n-gon, OB_1 its in-radius (r_1), and OA_1 its circum-radius (R_1).

Since B_1 bisects BC and $\triangle OB_1A_1$ is similar to OA_1C

$$\left.\begin{array}{ll} 2OB_1 = OB + OC = OB + OA, & \therefore\ 2r_1 = r + R \\ \text{and}\quad OA_1^2 = OB_1 \,.\, OC = OB_1 \,.\, OA, & \therefore\ R_1^2 = r_1 R \end{array}\right\}\quad \text{and } \therefore$$

$$2r_2 = r_1 + R_1,\quad 2r_3 = r_2 + R_2,\quad 2r_4 = \text{etc.},\quad 2r_m = r_{m-1} + R_{m-1} \quad \ldots \quad (a)$$

[redacted]

meter = 2 units, OB$(=r)$ vanishes, $R = OA = AB = \frac{1}{2}$ and Fig. 27 is modified to Fig. 28. Also, circumference of the circle $= 2 \mp 2\pi k$ $\therefore\ k = \dfrac{1}{\pi}$. Then, applying (*a*) and (*b*), we have

$$\left.\begin{array}{ll} r_3 = \cdot 314,\ 208,\ 718,\ 257,\ 8(7) & R_3 = \cdot 320,\ 364,\ 430,\ 968 \\ r_4 = \cdot 317,\ 286,\ 574,\ 613,\ \ldots & R_4 = \cdot 318,\ 821,\ 788,\ 7\ldots \\ r_5 = \cdot 318,\ 054,\ 181,\ 6(5) & R_5 = \cdot 318,\ 437,\ 75\ \ \ldots \end{array}\right\},\ldots(c)$$

Thus for the $2^5 . 2$-gon the radii agree to only 3 places. When m is large the following results will greatly reduce the labour of finding k.

2. OA_2 bisects $\angle A_1OC_1$, $\therefore$ A_1C_1 bisects $\angle CA_1B_1$ and $CC_1 > 2B_1B_2$. Thus $BB_1 > 4B_1B_2$, $B_1B_2 > 4B_2B_3$, etc.,

$$\text{and } BB_1 + B_1B_2 + \text{ etc.}, > 4(B_1B_2 + B_2B_3 + \text{etc.}),$$

i.e. $$BK > 4B_1K \text{ or } k - r > 4(k - r_1)$$

$\therefore$ finally $k - r_{m-1} > 4(k - r_m)$ and for a close value

$$r_m < k < \tfrac{1}{3}(4r_m - r_{m-1}), \quad \ldots \quad \ldots \quad \ldots \quad (d)$$

Thus by (*c*) $k = \frac{1}{3}(4r_6 - r_5) = \frac{1}{3}(2R_5 + r_5) = \cdot 318,\ 309,\ 89.$

FIGURE 28.

To find a similar relation between the circum-radii I draw $A_2D \parallel A_1C$ and A_2E bisecting $\angle C_1A_2D$. The four adjacent acute angles at A_2 are equal $\therefore$ $C_2C_1 < C_1E < ED$, CD or $C_1D > 2C_1C_2$. Thus $CC_1 > 4C_1C_2$, $C_1C_2 > 4C_2C_3$, etc., and, as before, CK or $R - k > 4C_1K$ or $4(R_1 - k)$. Finally $R_{m-1} - k > 4(R_m - k)$ and for a close value

$$R_m > k > \tfrac{1}{3}(4R_m - R_{m-1}), \quad \ldots \quad \ldots \quad \ldots \quad (e)$$

Thus by (*c*) without using r_6

$$\tfrac{1}{3}(4R_5 - R_4) = \cdot 318,\ 309,\ 74,\ \ \tfrac{1}{3}(4r_5 - r_4) = \cdot 318,\ 310,\ 05$$

$$\therefore \quad k = \text{arith. mean} = \cdot 318,\ 309,\ 89 \text{ as above.}$$

When r_m and R_m agree to p places $p - 1$ more can be found correctly by treating the new circum-radii as if they were in-radii.

Hence a third method of contraction when k has to be found to a large number of decimals. Calling the radii (after R_m) a_1, a_2, $a_3 \ldots a_x$ we shall have

$$\left.\begin{aligned} 2a_3 &= a_1 + a_2 \\ 2a_4 &= a_2 + a_3 \\ &\text{etc.} \\ 2a_x &= a_{x-2} + a_{x-1} \end{aligned}\right\} \quad \begin{aligned} &\therefore \text{ adding these } x-2 \text{ equations we get} \\ &\qquad a_1 + 2a_2 = 2a_x + a_{x-1} \\ &\qquad\qquad\quad = 3a_x \qquad \text{very nearly} \end{aligned}$$

$$\therefore \; k = \tfrac{1}{3}(r_m + 2R_m), \quad \ldots \quad \ldots \quad \ldots \quad {}^*(f)$$

without finding the $x-3$ intervening terms.

3. A fourth contraction may be derived from (d), thus:

$$4(k - r_m) < k - r_{m-1} \text{ gives } u = \tfrac{1}{3}(4r_m - r_{m-1})$$
$$4(k - r_{m-1}) < k - r_{m-2} \quad ,, \quad v = \tfrac{1}{3}(4r_{m-1} - r_{m-2})$$

and $\qquad 16(k-u) < k - v$

$\therefore$ for a close value $k = \frac{1}{15}(16u - v)$, $\ldots \quad \ldots \quad \ldots \quad \ldots$ (g)

Thus, using (c)

$$u = \tfrac{1}{3}(4r_5 - r_4) = \cdot 318,\ 310,\ 050,\ 7$$
$$v = \tfrac{1}{3}(4r_4 - r_3) = \cdot 318,\ 312,\ 526,\ 7$$

$\therefore$ by (g)

[redacted]

On the use of the Hyperbolic Sine and Cosine in connection with the Hyperbola.

By LAWRENCE CRAWFORD, M.A., B.Sc.

The excentric angle notation in the ellipse is extremely useful, and in part we can replace it by the hyperbolic sine and cosine in connection with the hyperbola.

Take the hyperbola $x^2/a^2 - y^2/b^2 = 1$, then the coordinates of any point on it may be written $a\cosh\phi$, $b\sinh\phi$, for $\cosh^2\phi - \sinh^2\phi = 1$. The objection to its use in all cases is that the hyperbolic cosine of an angle is always positive, so that $(a\cosh\phi, b\sinh\phi)$ can only represent any point on the branch on the positive side of the axis of y, for any point on the other branch we must take its coordinates as $(-a\cosh\phi, b\sinh\phi)$.

FIGURE 28.

Take up the discussion of conjugate diameters in this notation.

Take a series of parallel chords, joining, first, points on the same branch, and let QQ′ be one of them, to find the locus of their middle points.

The line joining the points $(x_1y_1)(x_2y_2)$ on the hyperbola has the equation

$$\frac{x(x_1+x_2)}{a^2} - \frac{y(y_1+y_2)}{b^2} = 1 + \frac{x_1x_2}{a^2} - \frac{y_1y_2}{b^2}$$

∴ if we take Q as the point $(a\cosh\alpha, b\sinh\alpha)$, Q′ as $(a\cosh\beta, b\sinh\beta)$ the equation of QQ′ is

$$\frac{x}{a}(\cosh\alpha + \cosh\beta) - \frac{y}{b}(\sinh\alpha + \sinh\beta) = 1 + \cosh\alpha\cosh\beta - \sinh\alpha\sinh\beta$$

i.e. $$2\frac{x}{a}\cosh\frac{\alpha+\beta}{2}\cosh\frac{\alpha-\beta}{2} - 2\frac{y}{b}\sinh\frac{\alpha+\beta}{2}\cosh\frac{\alpha-\beta}{2} = 1 + \cosh(\alpha-\beta)$$
$$= 2\cosh^2\frac{\alpha-\beta}{2}$$

i.e. $$\frac{x}{a}\cosh\frac{\alpha+\beta}{2} - \frac{y}{b}\sinh\frac{\alpha+\beta}{2} = \cosh\frac{\alpha-\beta}{2}.$$

If then we are to have a series of parallel chords we must join points, parameters α and β, such that

$$\frac{b}{a}\coth\frac{\alpha+\beta}{2} \text{ is constant.}$$

i.e. such that $\alpha+\beta$ is constant.

Join then points, parameters $\lambda+\mu$ and $\lambda-\mu$, keeping λ constant and varying μ, then we shall get a series of parallel chords, the gradient of which is

$$\frac{b}{a}\coth\frac{\lambda+\mu+\lambda-\mu}{2} \quad i.e. \ \frac{b}{a}\coth\lambda.$$

Now the middle point of the line joining the points,

$$(a\cosh\overline{\lambda+\mu},\ b\sinh\overline{\lambda+\mu}) \text{ and } (a\cosh\overline{\lambda-\mu},\ b\sinh\overline{\lambda-\mu})$$

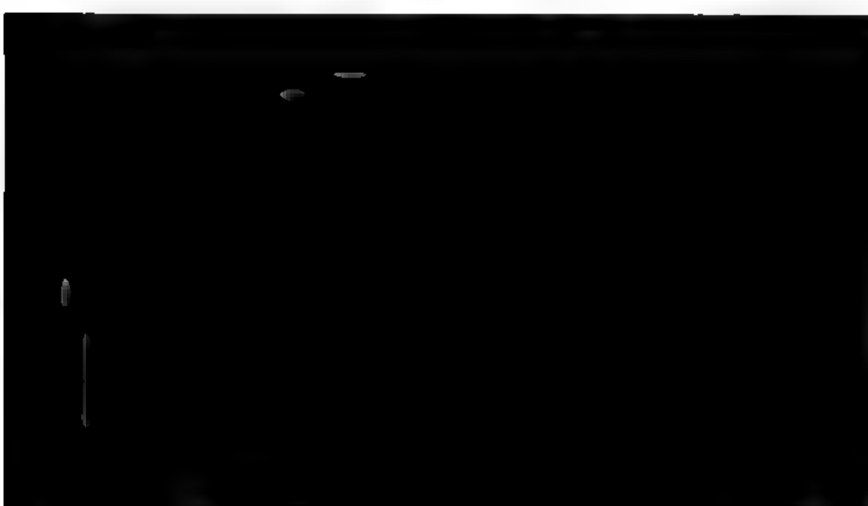
$$\text{is } \{\tfrac{1}{2}a(\cosh\overline{\lambda+\mu}+\cosh\overline{\lambda-\mu}),\ \tfrac{1}{2}b(\sinh\overline{\lambda+\mu}+\sinh\overline{\lambda-\mu})\}$$

i.e. $$(a\cosh\lambda\cosh\mu,\ b\sinh\lambda\cosh\mu)$$

$\therefore$ this point lies on the line $y=\frac{b}{a}\tanh\lambda\,.\,x$

$\therefore$ the locus of the middle points of this series of parallel chords, gradient $\frac{b}{a}\coth\lambda$, is the line $y=\frac{b}{a}\tanh\lambda\,.\,x$

Let us draw now a series of chords parallel to this latter line, $y=\frac{b}{a}\tanh\lambda\,.\,x$, and find locus of middle points of them.



The gradient of this line is $\frac{b}{a}\tanh\frac{\gamma-\delta}{2}$, and we wish it to be $\frac{b}{a}\tanh\lambda$, so join points, parameters γ and δ, so that $\gamma=\nu+\lambda$, $\delta=\nu-\lambda$ and keep λ constant but vary ν, then we get a series of parallel chords, gradient $\frac{b}{a}\tanh\lambda$.

The middle point of the line joining the points

$$(a\cosh\overline{\nu+\lambda},\ b\sinh\overline{\nu+\lambda})\quad(-a\cosh\overline{\nu-\lambda},\ b\sinh\overline{\nu-\lambda})$$

is

$$(a\sinh\nu\sinh\lambda,\ b\sinh\nu\cosh\lambda)$$

$\therefore$ this point lies on the line $y=\frac{b}{a}\coth\lambda\,.\,x$, which is therefore the locus of the middle points of this series of parallel chords, but this line is parallel to our original series of parallel chords,

$\therefore$ we have the lines $y=\frac{b}{a}\tanh\lambda\,.\,x$, $y=\frac{b}{a}\coth\lambda\,.\,x$ bisect each chords parallel to the other.

Thus the product of the gradients of two conjugate diameters is

$$\frac{b^2}{a^2}.$$

The line $y=\frac{b}{a}\tanh\lambda\,.\,x$ cuts the original hyperbola in P, which is the point $(a\cosh\lambda,\ b\sinh\lambda)$, while $y=\frac{b}{a}\coth\lambda\,.\,x$ cuts the conjugate hyperbola, $x^2/a^2-y^2/b^2=-1$ in the point $(a\sinh\lambda,\ b\cosh\lambda)$, say D.

These simple expressions for the coordinates of P and D give readily the theorems for example that $CP^2-CD^2=a^2-b^2$, that PD is bisected by the asymptote, and that tangents at P and D to the original and conjugate hyperbola respectively meet on the same asymptote.

Taking also again Q as the point $(a\cosh\overline{\lambda+\mu},\ b\sinh\overline{\lambda+\mu})$, and Q′ as $(a\cosh\overline{\lambda-\mu},\ b\sinh\overline{\lambda-\mu})$, and V as the middle point of QQ′, we easily prove

$$QV^2 : CV^2-CP^2 :: CD^2 : CP^2$$

which gives the equation of the hyperbola referred to the two conjugate diameters CP, CD as axes.

Seventh Meeting, May 10th, 1895.

WM. PEDDIE, Esq., M.A., D.Sc., in the Chair.

Proof of a Theorem in Conics.

By R. F. MUIRHEAD, M.A.

I.

In text books of Plane Coordinate Geometry, two methods are usually given for investigating the condition that the general equation of the second degree:

$$\phi \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

may represent a pair of real or imaginary straight lines.

The first is by identifying ϕ with the product of two linear factors, say $\lambda\lambda' \equiv (lx + my + nz)(l'x + m'y + n'z)$.

[redacted]

The object of this Note is to point out a short way of performing the elimination required in the former method, by forming the determinant which is the product of the two zero determinants

$$\begin{vmatrix} l, & l', & o \\ m, & m', & o \\ n, & n', & o \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} l', & l, & o \\ m', & m, & o \\ n', & n, & o \end{vmatrix}$$

The product is the symmetrical determinant

$$\begin{vmatrix} ll' + l'l, & lm' + l'm, & ln' + l'n \\ ml' + m'l, & mm' + m'm, & mn' + m'n \\ nl' + n'l, & nm' + n'm, & nn' + n'n \end{vmatrix}$$

which is of course identically equal to zero.

But if ϕ is identical with $\lambda\lambda'$ the determinant is obviously the same as

$$8 \times \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

Thus the discriminant of ϕ is zero if ϕ represents a pair of straight lines.

Of course $\lambda\lambda' = 0$ is the standard form when we have a pair of *real* straight lines; and can only represent an *imaginary* pair when some of the coefficients are imaginary. The standard form for a pair of imaginary lines (or point-ellipse) would be $\lambda^2 + \lambda'^2 = 0$, where $\lambda \equiv lx + my + nz$, etc.

In this case the identification with ϕ gives

$$a = l^2 + l'^2, \; f = mn + m'n', \text{ etc., etc.}$$

And the elimination of l, m, n, l', m', n' can here be performed by squaring the zero determinant

$$\begin{vmatrix} l & l' & o \\ m & m' & o \\ n & n' & o \end{vmatrix}$$

and substituting a for $l^2 + l'^2$, f for $mn + m'n'$, etc., in the result.

II.

It occurred to me recently that this method of getting the condition *discriminant* $=0$ by multiplying two determinants, might be capable of application to discuss the discriminant in the general case. I have only had leisure to make a beginning in this direction, and none to look up the literature of the subject; but the following results seem interesting, and are new to me.

Suppose the general expression ϕ put into the form

$$p\lambda^2+p'\lambda'^2+p''\lambda''^2,$$

where $pp'p''$ are constants and $\lambda\equiv lx+my+nz$, etc.; thus we have

$$a=pl^2+p'l'^2+p''l''^2,\ f=pmn+p'm'n'+p''m''n'',\ \text{etc., etc.}$$

and the discriminant $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$ is obviously

the product $\begin{vmatrix} l & l' & l'' \\ m & m' & m'' \\ n & n' & n'' \end{vmatrix} \times \begin{vmatrix} pl, & p'l', & p''l'' \\ pm, & p'm', & p''m'' \\ pn, & p'n', & p''n'' \end{vmatrix}$

which may be written

[redacted]

there are 8 independent ratios between the coefficients of the latter expression, and only 5 in ϕ.

Again, it appears that the discriminant may vanish in virtue of p'' being zero, in which case the value of λ'' might be anything whatever; in fact, it seems that in such a case, while two sides of a self-conjugate triangle must pass through the centre of the conic, the position of the third is quite indeterminate, a result which is obvious also from the geometrical point of view.

Theorems in the Products of Related Quantities.

By F. H. Jackson, M.A.

§ 1. Let $(x)_n$ denote the function

$$\underset{\kappa=\infty}{\mathrm{L}} \frac{(x-n+1)(x-n+2)\cdots(x-n+\kappa)}{(x+1)(x+2)\quad\cdots\quad(x+\kappa)}\cdot\kappa^n$$

then

$$\frac{(x+r)_n}{\underline{|r}\,\underline{|0}} - \frac{(x+r-1)_n}{\underline{|r-1}\,\underline{|1}} + \frac{(x+r-2)_n}{\underline{|r-2}\,\underline{|2}} - \cdots\cdots$$

$$\cdots + (-1)^r \frac{(x)_n}{\underline{|0}\,\underline{|r}} = \frac{n\cdot n-1\cdot n-2 \cdots n-r+1}{\underline{|r}}(x)_{n-r} \quad \ldots \ (1)$$

In Gamma Functions the above may be written.

$$\frac{\Gamma(x)}{\Gamma(x-n)} - {}_rC_1\frac{\Gamma(x-1)}{\Gamma(x-n-1)} + {}_rC_2\frac{\Gamma(x-2)}{\Gamma(x-n-2)} - \cdots\cdots$$

$$\cdots + (-1)^r\frac{\Gamma(x-r)}{\Gamma(x-n-r)} = (n)_r\frac{\Gamma(x-r)}{\Gamma(x-n)} \quad \ldots \ (2)$$

By using the theorem (1) I shall obtain a purely algebraical proof [redacted]

§2. **A fundamental property of the function $(x)_n$ is**

$$(x)_n \times (x-n)_m = (x)_{n+m}$$

whence we get $(x)_{n-r} \times (x-n+r)_s \times (x+r-s)_{r-s} = (x+r-s)_n$

Now the $(s+1)^{th}$ **term on the left side of** (1) $= (-1)^s \dfrac{(x+r-s)_n}{\lfloor r-s \, \lfloor s}$

which may be written

$$(-1)^s \frac{(x)_{n-r}(x-n+r)_s \cdot (x+r-s)_{r-s}}{\lfloor r-s \, \lfloor s}$$

Since $(x+r-s)_{r-s}$ —when r and s are both integers—may be written in the form

$$(x+1)(x+2)(x+3)\cdots(x+r-s) = (-1)^{r-s}(-x-1)_{r-s}$$

$\therefore$ **the** $(s+1)^{th}$ **term** $= (-1)^r \dfrac{(x)_{n-r}(x-n+r)\,(-x-1)_{r-s}}{\lfloor r-s \, \lfloor s}$

The expression on the left side of (1) may be written

$$(-1)^r \frac{(x)_{n-r}}{\lfloor r} \left\{ (-x-1)_r + \frac{\lfloor r}{\lfloor r-1 \, \lfloor 1}(-x-1)_{r-1}(x-n+r)_1 \right.$$
$$\left. + \frac{\lfloor r}{\lfloor r-2 \, \lfloor 2}(-x-1)_{r-2}(x-n+r)_2 + \cdots + (-1)^r(x-n+r)_r \right\} \quad (4)$$

By Vandermonde's theorem* the expression with the large bracket

$$= (-x-1+x-n+r)_r = (r-n-1)_r$$

Expression (4) becomes

$$(-1)^r \frac{(x)_{n-r}}{\lfloor r}(-n-1)_r = \frac{n \cdot n-1 \cdot n-2 \cdots n-r+1}{\lfloor r}(x)_{n-r}$$

which proves theorem (1).

---

* See §7.

§ 3. Now $$(x)_n = \frac{\Gamma(x+1)}{\Gamma(x-n+1)}$$

$$\because \quad \Gamma(x+1) = \underset{\kappa=\infty}{\mathrm{L}} \frac{1\cdot 2\cdot 3\cdot 4 \cdots \kappa}{(x+1)(x+2)\cdots(x+\kappa)}\kappa^x$$

Replacing $(\ )_n$ by Gamma Functions, the theorem (1), after multiplication throughout by $\underline{|r}$, becomes

$$\frac{\Gamma(x+r+1)}{\Gamma(x+r-n+1)} - {}_rC_1\frac{\Gamma(x+r)}{\Gamma(x+r-n)} + {}_rC_2\frac{\Gamma(x+r-1)}{\Gamma(x+r-n-1)} - \quad \cdots\cdots$$
$$\cdots\cdots + (-1)^r\frac{\Gamma(x+1)}{\Gamma(x-n+1)} = (n)_r\frac{\Gamma(x+1)}{\Gamma(x-n+r+1)} \qquad (5)$$

substitute y for $x+r+1$, then (5) becomes

$$\frac{\Gamma(y)}{\Gamma(y-n)} - {}_rC_1\frac{\Gamma(y-1)}{\Gamma(y-n-1)} + {}_rC_2\frac{\Gamma(y-2)}{\Gamma(y-n-2)} - \quad \cdots$$
$$\cdots\cdots + (-1)^r\frac{\Gamma(y-r)}{\Gamma(y-n-r)} = (n)_r\frac{\Gamma(y-r)}{\Gamma(y-n)}$$

Remembering that $\Gamma(y) = (y-1)\Gamma(y-1)$ on division throughout by $\frac{\Gamma(y)}{\Gamma(y-n)}$ we have

[redacted]

§ 4. To consider the expansion in general of $f(x+y)$ in the form $\qquad P_0+P_1(x)_1+P_2(x)_2+\cdots+P_r(x)_r+\cdots$
where $P_0\cdot P_1\cdot P_2\cdots$ are functions of y only or constants. Assume that $f(x+y)$ is capable of being expanded in a convergent series of the above form then

$$f(x+y)=P_0+P_1(x)_1+P_2(x)_2+\cdots+P_r(x)_r+\cdots$$

By giving x the values $0\cdot 1\cdot 2\cdot 3\cdots$ in succession we obtain the following equations to determine $P_0\cdot P_1\cdot P_2\cdots$

$$\begin{aligned}
f(y) &= P_0\\
f(y+1) &= P_0+P_1\\
f(y+2) &= P_0+2P_1+2\cdot 1\ P_2\\
&\ldots\ldots\ldots\ldots\\
f(y+r) &= P_0+r\cdot P_1+r\cdot r-1\ P_2+\ldots\ \underline{|r}\cdot P\\
&\ldots\ldots\ldots\ldots
\end{aligned}$$

From which we obtain

$$P_0=\frac{f(y)}{\underline{|0}\ \underline{|0}}$$

$$P_1=\frac{f(y+1)}{\underline{|1}\ \underline{|0}}-\frac{f(y)}{\underline{|0}\ \underline{|1}}$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$P_r=\frac{f(y+r)}{\underline{|r}\ \underline{|0}}-\frac{f(y+r-1)}{\underline{|r-1}\ \underline{|1}}+\frac{f(y+r-2)}{\underline{|r-2}\ \underline{|2}}-\cdots+(-1)^r\frac{f(y)}{\underline{|0}\ \underline{|r}}$$

$$\ldots\ldots\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

which is that

$$f(x+y)=\sum_{r=0}^{r=\infty}\sum_{s=0}^{s=r}(-1)^s\frac{f(y+r-s)}{\underline{|r-s}\ \underline{|s}}(x)_r \qquad \ldots \qquad \ldots \qquad (7)$$

subject to the convergence of the series.

§ 5. **The expansion of $(x+y)_n$, n being unrestricted.**

The coefficient of x_r will be

$$\mathrm{P}_r \equiv \frac{(y+r)_n}{\underline{|n}\,\underline{|0}} - \frac{(y+r-1)_n}{\underline{|n-1}\,\underline{|1}} + \frac{(y+r-2)_n}{\underline{|n-2}\,\underline{|2}} - \cdots$$

$$\cdots + (-1)^r \frac{(y)_n}{\underline{|0}\,\underline{|n}} \equiv \frac{n \cdot n-1 \cdots n-r+1}{\underline{|r}} y_{n-r}$$

(by Theorem (1)).

$$\therefore \quad (x+y)_n = y_n + ny_{n-1}x_1 + \frac{n \cdot n-1}{\underline{|2}} y_{n-2}x_2 + \cdots$$

$$+ \frac{n \cdot n-1 \cdot\ \ x-r+1}{\underline{|r}} y_{n-r}x_r + \cdots \quad (8)$$

This is the generalised form of Vandermonde's Theorem; the proof depends, as will be seen on reference to § 2, No. 4, on Vandermonde's Theorem for positive integral values of the suffix.

To expand a^x in a series of form (7)

we have $\quad \mathrm{P}_r = \dfrac{a^r}{\underline{r}\ \underline{0}} - \dfrac{a^{r-1}}{\underline{|r-1}\,\underline{|1}} + \cdots + (-1)^r \dfrac{a^0}{\underline{|0}\,\underline{|r}} = \dfrac{(a-1)^r}{\underline{|r}}$

$$\therefore \quad a^x = 1 + (a-1)(x)_1 + \frac{(a-1)^2}{\underline{|2}}(x)_2 + \cdots + \frac{(a-1)^r}{\underline{r}}(x)_r + \cdots$$

[redacted]

§ 6. Writing

$$(x+y)_n = (y)_n + n \cdot (y)_{n-1}(x)_1 + \frac{n \cdot n-1}{\underline{2}\,\lfloor r}(y)_{n-2}(x)_2 + \cdots$$

$$\cdots\cdots + \frac{n \cdot n-1 \cdots n-r+1}{\lfloor r}(y)_{n-r}(x)_r + \cdots\cdots$$

divide both sides by $(y)_n$.

Then $\dfrac{(x+y)_n}{(y)_n} = 1 + n \cdot \dfrac{(y)_{n-1}(x)_1}{(y)_n} + \dfrac{n \cdot n-1}{\lfloor 2 \;\lfloor r} \dfrac{(y)_{n-2}(x)_2}{(y)_n} + \cdots$

Now it is easily seen that $\dfrac{(y)_{n-1}}{(y)_n} = \dfrac{1}{y-n+1}$

$$\cdots\cdots\cdots\cdots$$

$$\frac{(y)_{n-r}}{(y)_n} = \frac{1}{(y-n+1)_r}$$

and $\dfrac{(x+y)_n}{(y)_n} = \dfrac{\Pi(x+y)}{\Pi(x+y-n)} \cdot \dfrac{\Pi(y-n)}{\Pi(x)}$ where Π denotes Gauss's Π Function. Therefore

$$\frac{\Pi(x+y) \cdot \Pi(y-n)}{\Pi(x+y-n)\,\Pi(x)} = 1 + n \cdot \frac{(x)_1}{(y-n+1)_1}$$

$$+ \frac{n \cdot n-1}{2\,!} \frac{(x)_2}{(y-n+1)(y-n+2)} + \cdots + \frac{n \cdots n-r+1}{r\,!} \frac{(x)_r}{(y-n+1)_r} + \cdots$$

Replacing n by $-\alpha$, x by $-\beta$, and $y-n+1$ by γ we have

$$\frac{\Pi(\gamma-\alpha-\beta-1) \cdot \Pi(\gamma-1)}{\Pi(\gamma-\beta-1)\Pi(\gamma-\alpha-1)} = 1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} + \frac{\alpha \cdot \alpha+1 \cdot \beta \cdot \beta+1}{1 \cdot 2 \cdot \gamma \cdot \gamma+1} + \cdots \tag{9}$$

$$= F_1(\alpha, \beta, \gamma)$$

§ 7. If in § 2, result (4), we had assumed the truth of Vandermonde's Theorem for unrestricted values of the suffix, Theorems (1), (2), and (3) would have been proved for all values of r. Since we have proved Vandermonde's Theorem for unrestricted values of the suffix, the proofs of §§ 2 and 3 may be repeated with r unrestricted. The use of $(-1)^r$ in § 2 can easily be avoided. When r is unrestricted, $(r)_r$ must be used instead of $\lfloor r$.

Isogonals of a Triangle.

By J. S. Mackay, M.A., LL.D.

Definition.—*If two angles have the same vertex and the same bisector, the sides of either angle are isogonal* to each other with respect to the other angle.*

Thus the isogonal of AP with respect to $\angle$ BAC is the image of AP in the bisector of $\angle$ BAC. It is indifferent whether the bisector of the interior $\angle$ BAC be taken, or the bisector of the angle adjacent to it; the isogonal of AP remains the same.

It follows from the definition that

(1) The internal and the external bisectors of $\angle$ BAC are their own isogonals.

(2) The line joining the orthocentre of a triangle to any vertex is isogonal to the line joining the circumcentre to that

[redacted]

§ 1.

(a) *If P, Q be any two points taken on a pair of lines isogonal with respect to angle BAC, the distances of P from AB, AC are inversely proportional* to those of Q from AB, AC.*

FIGURE 29.

If the quadrilateral AQ_2QQ_1 be revolved through two right angles round the bisector of $\angle B$ as an axis, it will become homothetic to the quadrilateral AP_1PP_2; therefore

$$PP_1 : PP_2 = QQ_2 : QQ_1$$

(a′) *If P, Q be any two points and if the distances of P from AB, AC be inversely proportional to those of Q from AB, AC, then AP, AQ are isogonal with respect to $\angle BAC$.*

This may be proved indirectly.

(1) *The points P_1 Q_1 Q_2 P_2 are concyclic*†

Since P_1P_2 Q_1Q_2 are antiparallel with respect to $\angle BAC$; therefore P_1 P_2 Q_1 Q_2 are concyclic.

(2) *The centre of the circle $P_1Q_1Q_2P_2$ is the mid point of PQ.*

For the perpendicular to P_1Q_1 at its mid point goes through the centre of the circle; and this perpendicular bisects PQ.
So does the perpendicular to P_2Q_2 at its mid point.

(3) *P_1P_2 is perpendicular to AQ*
and Q_1Q_2 „ „ „ AP.

---

* Sir James Ivory in Leybourn's *Mathematical Repository*, new series, Vol. I., Part II., p. 19 (1806). The mode of proof is due to Professor Neuberg. See his excellent memoir on the Recent Geometry of the Triangle in Rouché and Comberousse's *Traité de Géométrie*, First Part, p. 438 (1891).

† This and the two following theorems are due to Steiner. See Gergonne's *Annales*, XIX., 37-64 (1828), or Steiner's *Gesammelte Werke*, I., 191-210 (1881). The proof given of (1) is Professor Neuberg's. See the reference in the preceding note.

For AP is a diameter of the circumcircle of AP_1P_2; therefore the isogonal of AP with respect to $\angle P_1AP_2$ is the perpendicular * from A to P_1P_2

(4) The circumcentre of either of the triangles AP_1P_2 AQ_1Q_2 and the orthocentre of the other are collinear with the point A.

(5) *Triangle PP_1P_2 is inversely similar † to QQ_2Q_1.*

This follows from the demonstration of § 1; or it may be thus proved:

$$\angle PP_1P_2 = \angle PAP_2 = \angle QAQ_1 = \angle QQ_2Q_1.$$

Similarly $$\angle PP_2P_1 = \angle QQ_1Q_2.$$

(6) *If PP_1 QQ_2 meet at D*
and PP_2 QQ_1 „ „ E,
then AD, AE are isogonals with respect to $\angle BAC$.

FIGURE 30.

Join P_1Q_2 P_2Q_1.

Since P_1Q_1 P_2Q_2 are concyclic,
therefore $\angle AQ_2P_1 = \angle AQ_1P_2$
therefore their complements are equal

[redacted]

§ 2.

(a) *If ABC be a triangle, and if AP, AQ be isogonal with respect to A, then* *

$$BP \cdot BQ : CP \cdot CQ = AB^2 : AC^2$$

FIGURE 31.

About APQ circumscribe a circle, cutting AB, AC in F, E; join FE.

Because	$\angle BAP = \angle CAQ$
therefore	arc FP = arc EQ
therefore	FE is parallel to BC
therefore	$AB : BF = AC : CE$
therefore	$AB^2 : AB \cdot BF = AC^2 : AC \cdot CE$
therefore	$AB^2 : BP \cdot BQ = AC^2 : CP \cdot CQ$.

A second demonstration will be found in C. Adams's *Die merkwürdigsten Eigenschaften des geradlinigen Dreiecks*, p. 1 (1846), and a third in Professor Fuhrmann's *Synthetische Beweise*, p. 94 (1890).

(a') *If ABC be a triangle and BC be divided at P and Q so that*

$$BP \cdot BQ : CP \cdot CQ = AB^2 : AC^2$$

then † *AP, AQ are isogonals with respect to A.*

This may be proved indirectly.

(1) If AQ be the internal or the external median from A, then $BQ = CQ$, and the theorem becomes ‡

$$BP : CP = AB^2 : AC^2.$$

---

* Pappus's *Mathematical Collection*, VI. 12. The same theorem differently stated is more than once proved in Book VII. among the lemmas which Pappus gives for Apollonius's treatise on *Determinate Section*. The proof in the text is taken from Pappus.

† In Pappus's *Mathematical Collection*, VI. 13, there is proved the theorem :

$$\text{If } BP \cdot BQ : CP \cdot CQ > AB^2 : AC^2$$

then $$\angle BAP > \angle CAQ.$$

‡ Adams (see the reference to him on this page) gives (1)-(4), (6), (8). His proof of (4) is different from that in the text.

(2) *If AQ be the internal or the external median from A and $\angle BAC$ be right, then AP is perpendicular to BC.*

FIGURES 32, 33.

Since $\qquad \angle ACB = \angle CAQ = \angle BAP$

therefore $\angle ACB + \angle CAP = \angle BAP + \angle CAP$

$= $ a right angle.

(3) If AP and AQ coincide, then AP is either the internal or the external bisector of $\angle A$, and the theorem becomes

$$BP^2 : CP^2 = AB^2 : AC^2$$

or
$$BP : CP = AB : AC$$

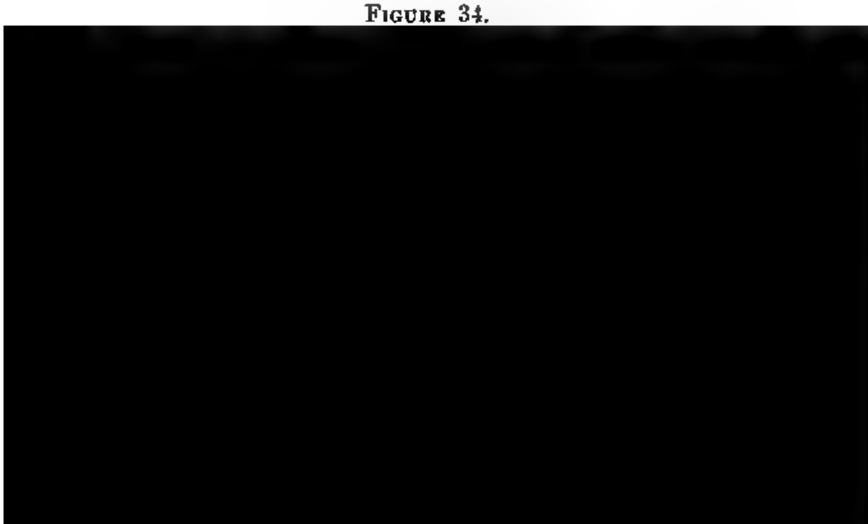
a known result, namely, Euclid VI. 3, or the cognate theorem.

(4) $$BP \cdot CP : BQ \cdot CQ = AP^2 : AQ^2.$$

This follows from the theorem of §2 by considering APQ as the triangle and AB, AC as the isogonals.

(5) *If AP, AQ which are isogonal with respect to $\angle BAC$ meet the circumcircle of ABC in R, S, then $AP \cdot AS = AQ \cdot AR$.*

FIGURE 34.



(8) If APR becomes the diameter of the circumcircle ABC then AQ becomes perpendicular to BC, and

$$AQ \cdot AR = AB \cdot AC,$$

a theorem of Brahmegupta's.

See Chasles's *Aperçu*, 2nd ed., pp. 420-447.

(9) If AP, AQ coincide, then AP becomes either the internal or the external bisector of $\angle$ A.
Hence in the first case

$$\begin{aligned} AB \cdot AC &= AP \cdot AS \\ &= AP \cdot PS + AP^2 \\ &= BP \cdot PC + AP^2; \end{aligned}$$

and in the second case

$$\begin{aligned} AB \cdot AC &= AP \cdot AS \\ &= AP \cdot PS - AP^2 \\ &= BP \cdot PC - AP^2. \end{aligned}$$

(10) *In triangle ABC, AP, AQ are isogonals with respect to A ;*
through B draw BE parallel to AP meeting CA in E ;
„ C „ CF „ „ AQ „ BA „ F;
*then EF is antiparallel * to BC with respect to A.*

FIGURE 35.

For $\angle ABE = \angle BAP = \angle CAQ = \angle ACF$;
therefore the points E, B, C, F are concyclic.

The same thing would happen if BE, CF were drawn parallel to AQ, AP.

(11) *In triangle ABC, AP, AQ are isogonal; from P and Q perpendiculars are drawn to BC ; these perpendiculars are intersected at D, E by a perpendicular to AB at B, and at D', E' by a perpendicular to AC at C. To prove* †

$$BD \cdot BE : CD' \cdot CE' = AB^4 : AC^4.$$

* Mr Emile Vigarié.

† Mr Emile Vigarié in the *Journal de Mathématiques Élémentaires*, 2nd series, IV. 224 (1885) says that this theorem was communicated to him by his friend Mr Th. Valiech.

FIGURE 36.

Draw AX perpendicular to BC.

The similar triangles BDP, BEQ, ABX give

$$BD : BP = AB : AX$$
$$BE : BQ = AB : AX$$

therefore
$$\frac{BD \cdot BE}{BP \cdot BQ} = \frac{AB^2}{AX^2}$$

Similarly
$$\frac{CD' \cdot CE'}{CP \cdot CQ} = \frac{AC^2}{AX^2}$$

therefore
$$\frac{BD \cdot BE}{CD' \cdot CE'} \cdot \frac{CP \cdot CQ}{BP \cdot BQ} = \frac{AB^2}{AC^2}$$

therefore
$$\frac{BD \cdot BE}{CD' \cdot CE'} \cdot \frac{AC^2}{AB^2} = \frac{AB^2}{AC^2}$$

(12) *If in* (11) *AQ be the median* from A, then* $BD : CD' = AB^2 : AC^2$.

FIGURE 36.

For $BE : BQ = AB : AX$

and $CE' : CQ = AC : AX$;

therefore $BE : CE' = AB : AC$,

whence the result follows.

FIGURE 37.

Let BO, BO′ be isogonals with respect to B
and CO, CO′ „ „ „ „ C;
then AO, AO′ are „ „ „ A.

Denote the distances of O from the sides by p_1 p_2 p_3 and those of O′ by q_1 q_2 q_3

Then $$p_1 q_1 = p_2 q_2 \text{ and } p_1 q_1 = p_3 q_3$$

therefore $$p_2 q_2 = p_3 q_3$$

therefore AO, AO′ are isogonals with respect to A.

Another demonstration will be found in C. Adams's *Eigenschaften des...Dreiecks*, pp. 7-8 (1846).

Points such as O, O′ determined by the intersection of pairs of isogonal lines will be called *isogonal points*, or simply *isogonals*, with respect to the triangle ABC.

They are sometimes* called *isogonally conjugate points*, or *isogonal conjugates*, but more frequently on the continent of Europe *inverse points* with respect to the triangle ABC.

The designation, *inverse points*, was suggested about the same time in Scotland and in France. See a paper read before the Royal Society of Edinburgh on 20th March 1865, by the Rev. Hugh Martin, and printed in their *Transactions*, xxiv. 37-52: and an article by Mr J. J. A. Mathieu in the *Nouvelles Annales*, 2nd series, IV. 393–407, 481-493, 529-537 (1865).

Perhaps the adoption of the nomenclature proposed by Mr G. de Longchamps in the *Journal de Mathématiques Élémentaires*, 2nd series, V. 109 (1886) would be advantageous.

(1) $\angle BOC + \angle BO'C = 180° + A.$

FIGURE 37.

For $$\angle \mathrm{BOC} = \mathrm{A} + \mathrm{ABO} + \mathrm{ACO},$$
$$= \mathrm{A} + \mathrm{CBO'} + \mathrm{BCO'},$$

and $$\angle \mathrm{BO'C} = \mathrm{A} + \mathrm{ABO'} + \mathrm{ACO'};$$

therefore $$\angle \mathrm{BOC} + \angle \mathrm{BO'C} = 2\mathrm{A} + \mathrm{B} + \mathrm{C},$$
$$= 180° + \mathrm{A}.$$

* Professor J. Neuberg's *Mémoire sur le Tétraèdre*, p. 10 (1884).

(2) In triangle ABC, AP_1 BP_2 CP_3 are concurrent at O, and their isogonals AQ_1 BQ_2 CQ_3 are concurrent at O′.

FIGURE 36.

Suppose BP_2 BQ_2 to form one straight line
and CP_3 CQ_3 „ „ „ „ „ ;
then the points O O′ coincide.*

There are four cases.

(*a*) If BP_2 CP_3 bisect the interior angles B, C, then AP_1 bisects the interior angle A.

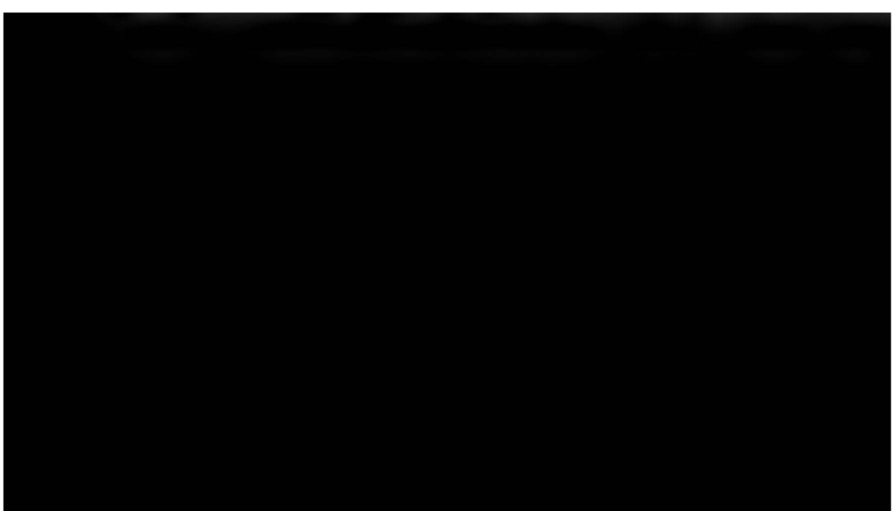
(*b*) If BP_2 CP_3 bisect the exterior angles B, C, then AP_1 bisects the interior angle A.

(*c*) If BP_2 bisects the interior angle B
and CP_3 „ „ exterior „ C,
then AP_1 „ „ exterior „ A.

(*d*) If BP_2 bisects the exterior angle B
and CP_3 „ „ interior „ C,
then AP_1 „ „ exterior „ A.

Hence the six bisectors of the angles of a triangle meet three by three in four points.

FIGURE 36.



§ 4.

Positions of two isogonal points with reference to a triangle.

(1) Any point on a side has for isogonal point the opposite vertex.

(2) A vertex has for isogonal point any point on the opposite side.

(3) A point inside the triangle has its isogonal point also inside the triangle.

(4) If a point be outside the triangle and situated in the angle vertically opposite to $\angle$ BAC, for example, its isogonal point will be outside the triangle and situated in that segment of the circumcircle (remote from A) cut off by BC.

(5) If a point be outside the circumcircle and situated within the angle BAC, for example, its isogonal point will be outside the circumcircle and situated within the same angle.

(6) If a point be on the circumference of the circumcircle, its isogonal point will be at infinity.

The truth of these statements,* which are not quite obvious, may be ascertained by the construction of a few figures. Of the last statement the following proof may be given :—

FIGURE 39.

If AD, BE, CF, be three parallel lines drawn through the vertices of a triangle ABC, their three isogonals will be concurrent at a point on the circumference of the circumcircle.†

Because AD, BE, CF are parallel,

therefore arc AE = arc BD, arc BC = arc EF.

Make arc CP equal to arc BD; join AP, BP, CP.

* They are all given by Mr J. J. A. Mathieu in *Nouvelles Annales*, 2nd series, IV. 403 (1865).

† Professor Eugenio Beltrami in *Memorie del l'Accademia delle Scienze del Istituto di Bologna*, 2nd series, II., 383 (1863).

Since arc CP = arc BD,
therefore ∠ CAP = ∠ BAD,
and AP is isogonal to AD.

Since arc CP = arc AE,
therefore ∠ CBP = ∠ ABE,
and BP is isogonal to BE.

Since arc BC = arc EF, arc CP = arc AE,
therefore arc BP = arc AF;
therefore ∠ BCP = ∠ ACF,
and CP is isogonal to CF.

Hence, if P be a point on the circumcircle of ABC, the point isogonal to it is the point of concurrency of AD, BE, CF.

(1) AD is perpendicular* to the Wallace line P (ABC).

This follows from § 1, (3).

§ 5.

*If three angular transversals cut the opposite sides in three collinear points, their isogonals will also cut the opposite sides in three collinear points.**

FIGURE 40.

[redacted]

Now $$\frac{BD \cdot CE \cdot AF}{CD \cdot AE \cdot BF} = 1;$$

therefore $$\frac{BD' \cdot CE' \cdot AF'}{CD' \cdot AE' \cdot BF'} = 1;$$

therefore D′, E′, F′ are collinear.

§ 6.

If O be any point in the plane of triangle ABC, and AO BO CO meet the circumcircle in A_1 B_1 C_1 and D E F be the projections of O on BC CA AB the triangles $A_1B_1C_1$ DEF are directly similar, and the point O of triangle DEF corresponds to that point of $A_1B_1C_1$ which is isogonal to O.*

FIGURE 41.

For the points O F B D are concyclic;

therefore $$\angle FDO = \angle FBO$$
$$= \angle B_1A_1O.$$

Similarly $$\angle EDO = \angle C_1A_1O.$$

The demonstration may be easily seen to apply to the more general case where A_1 B_1 C_1 are taken inverse to O with any other constant of inversion.*

(1) If O be the orthocentre of ABC, it must be the incentre or an excentre of DEF, and therefore the incentre or an excentre of $A_1B_1C_1$.

(2) If O be the circumcentre of ABC, it must be the orthocentre of DEF, and therefore the circumcentre of $A_1B_1C_1$.

(3) If O be the incentre of ABC, it must be the circumcentre of DEF, and therefore the orthocentre of $A_1B_1C_1$.

(4) If O be an excentre of ABC, it must be the circumcentre of DEF, and therefore the orthocentre of $A_1B_1C_1$.

---

* Mr E. M. Langley and Professor Neuberg. (1)—(4) are Mr Langley's. See the *Seventeenth General Report of the Association for the Improvement of Geometrical Teaching*, p. 45 (1891.)

§ 7

*If two points be isogonal with respect to a triangle their six projections on the sides of the triangle are concyclic.**

FIGURE 42.

Let O, O′ be isogonal with respect to ABC, and let D, E, F, D′, E′, F′ be their projections on the sides BC, CA, AB.

Then EF is antiparallel to E′F′ with respect to A ;

therefore E,E′ F,F′, are concyclic.
Similarly F,F′ D,D′ „ „ ,
and D,D′ E,E′ „ „ ;

therefore the six points are concyclic.

§ 8

If O O′ be isogonal points with respect to ABC, and D E F D′E′F′ be their respective projections on BC CA AB, then

[redacted]

Eighth Meeting, Friday, June 14th, 1895.

JOHN M'COWAN, Esq., M.A., D.Sc., President, in the Chair.

A Summary of the Theory of the Refraction of their approximately Axial Pencils through a Series of Media bounded by coaxial Spherical Surfaces, with Applications to a Photographic Triplet, &c.

By PROFESSOR CHRYSTAL.

[*The Paper will be published in the next Volume.*]

On a Diophantine Equation.

By R. F. DAVIS, M.A.

In the consideration of Question 12612 appearing in the *Educational Times* for January of this year, proposed by the Rev. Dr. Haughton, F.R.S., of Trinity College, Dublin, the following Diophantine Equation suggests itself:

What values of x make $8x^3-8x+16=\square$?

Since it may be written $8x(x^2-1)+16=\square$ it is obvious that $x=0, \pm 1$ are solutions. Also that $x=2$ is a solution. Moreover $x=-\dfrac{3}{2}$ when substituted gives $-27+12+16=1$ and is therefore a solution,—marking approximately a limit to the negative root.

I. Put $8x^3-8x+16=(px^2+x-4)^2$; then after reduction and division by x^2, we have

$$p^1x^2-2x(4-p)+1-8p=0 \qquad \ldots \qquad \ldots \qquad \ldots \qquad (A)$$

It will be found that the roots of this equation are real and rational when $\qquad 8p^3-8p+16=\square$

which is the same Diophantine Equation as that with which we started.

Hence the values of x obtained by experiment may be used for p in the equation (A) with the certainty of obtaining one or more fresh solutions.

Thus put $p=0$ and we get $\quad -8x+1=0 \quad x=\frac{1}{8}$

" " $p=1$ " " " $x^2-6x+7=0 \quad x=-1$ or 7

" " $p=1$ " " " $x^2-10x+9=0 \quad x=+1$ or 9

" " $p=2$ " " " $4x^2-4x+15=0 \quad x=\frac{5}{2}, \frac{3}{2},$

all depending on the fact that if one root of a quadratic equation be real and rational, so is the other root.

II. The equation (A) may be written

$$\begin{aligned}(px+1)^2 &= 8(x+p)\\ &= 16a^2, \quad \text{say};\end{aligned}$$

whence $\qquad px+1=4a, \quad$ and $\quad x+p=2a^2.$

Thus $\qquad x(2a^2-x)-4a+1=0$

$$x^2-2a^2x+4a-1=0 \qquad \ldots \qquad \ldots \qquad \text{(B)}$$

[redacted]

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11. **Journal de Mathématiques Élémentaires.**
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Nos. 1-19. 1894-95.

12. **Journal de Mathématiques Élémentaires.**
Publié sous la direction de M. de Longchamps.
Nos. 8-12. 1894.
Nos. 1-7. 1895.

13. **Journal de Mathématiques Spéciales.**
Publié sous la direction de M. de Longchamps.
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Nos. 1-7. 1895.

14. **Répertoire Bibliographique des Sciences Mathématiques. Première** Série : Fiches 1 à 100.

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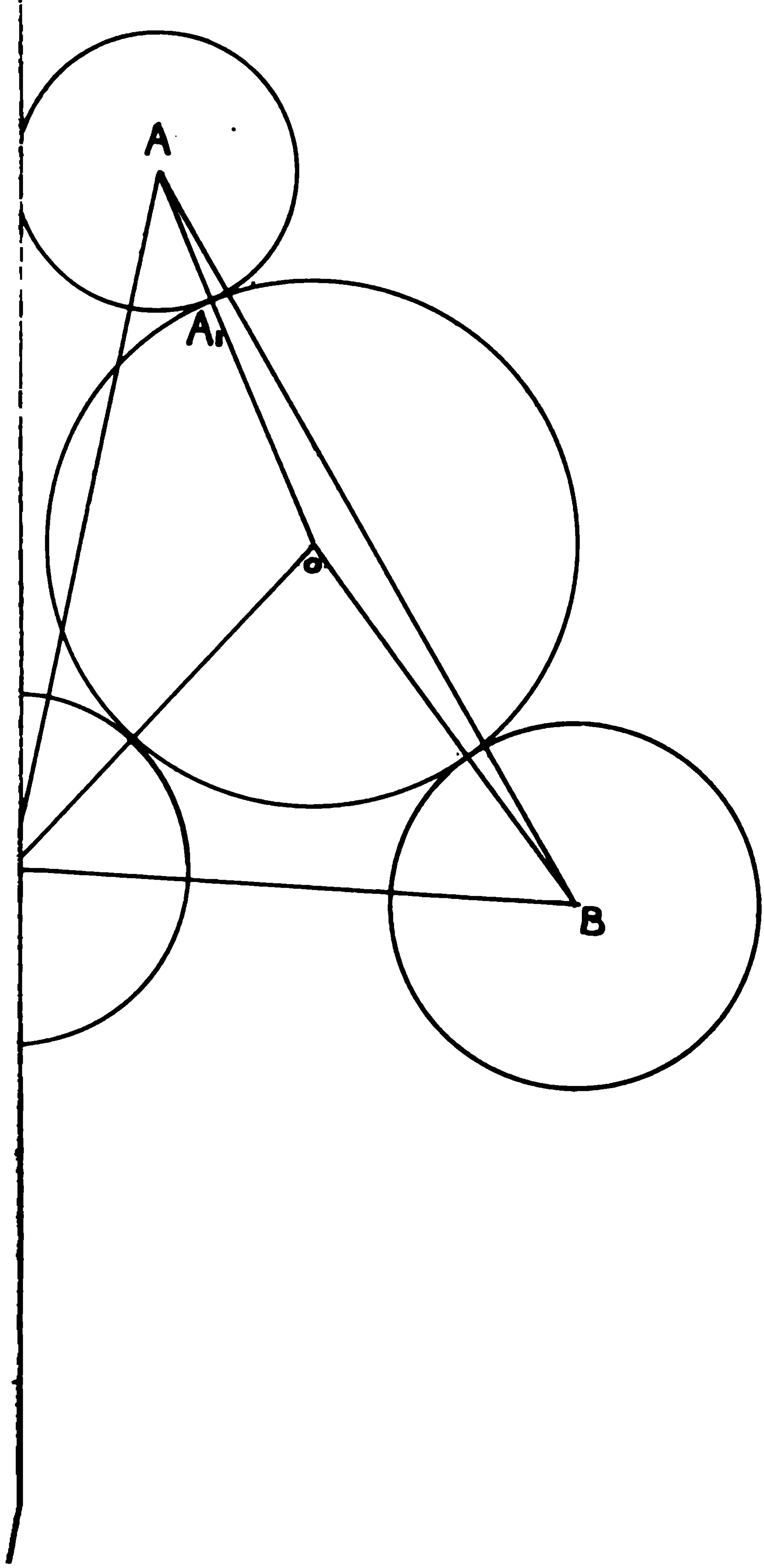
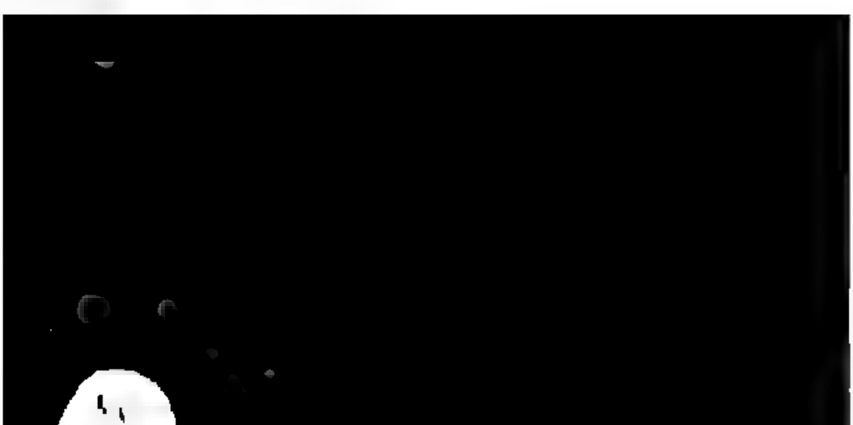
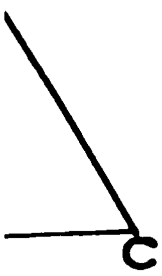
FIG. 1.



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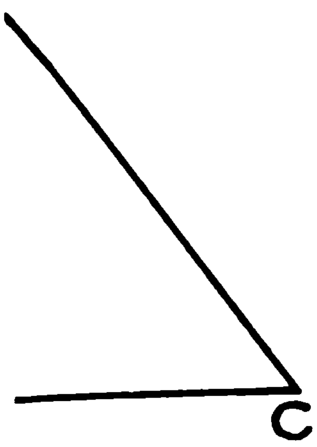
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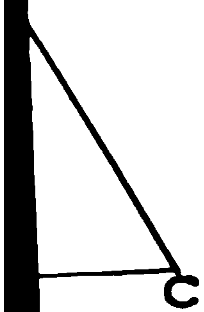
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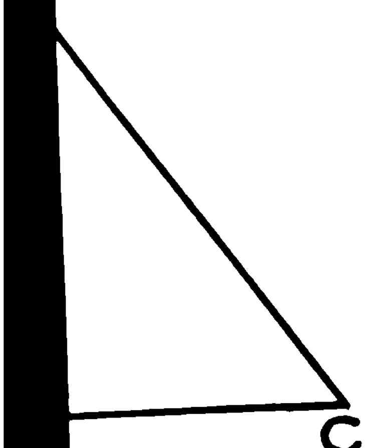
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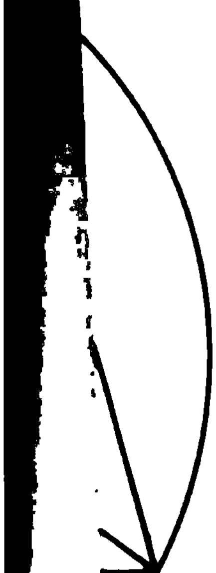
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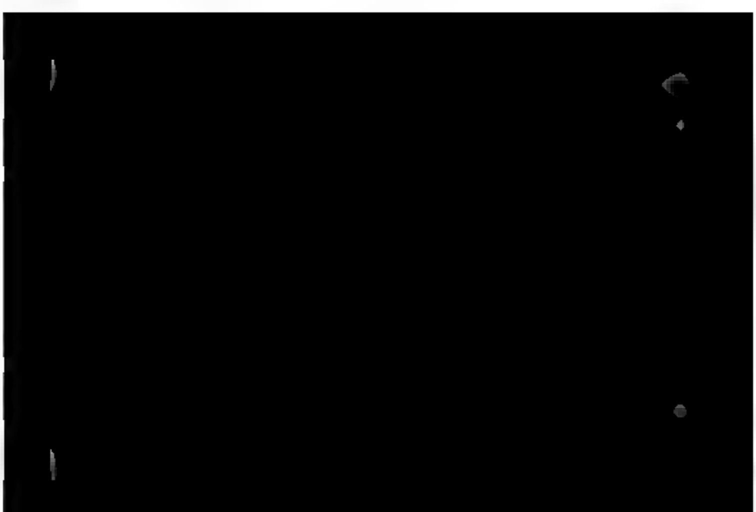
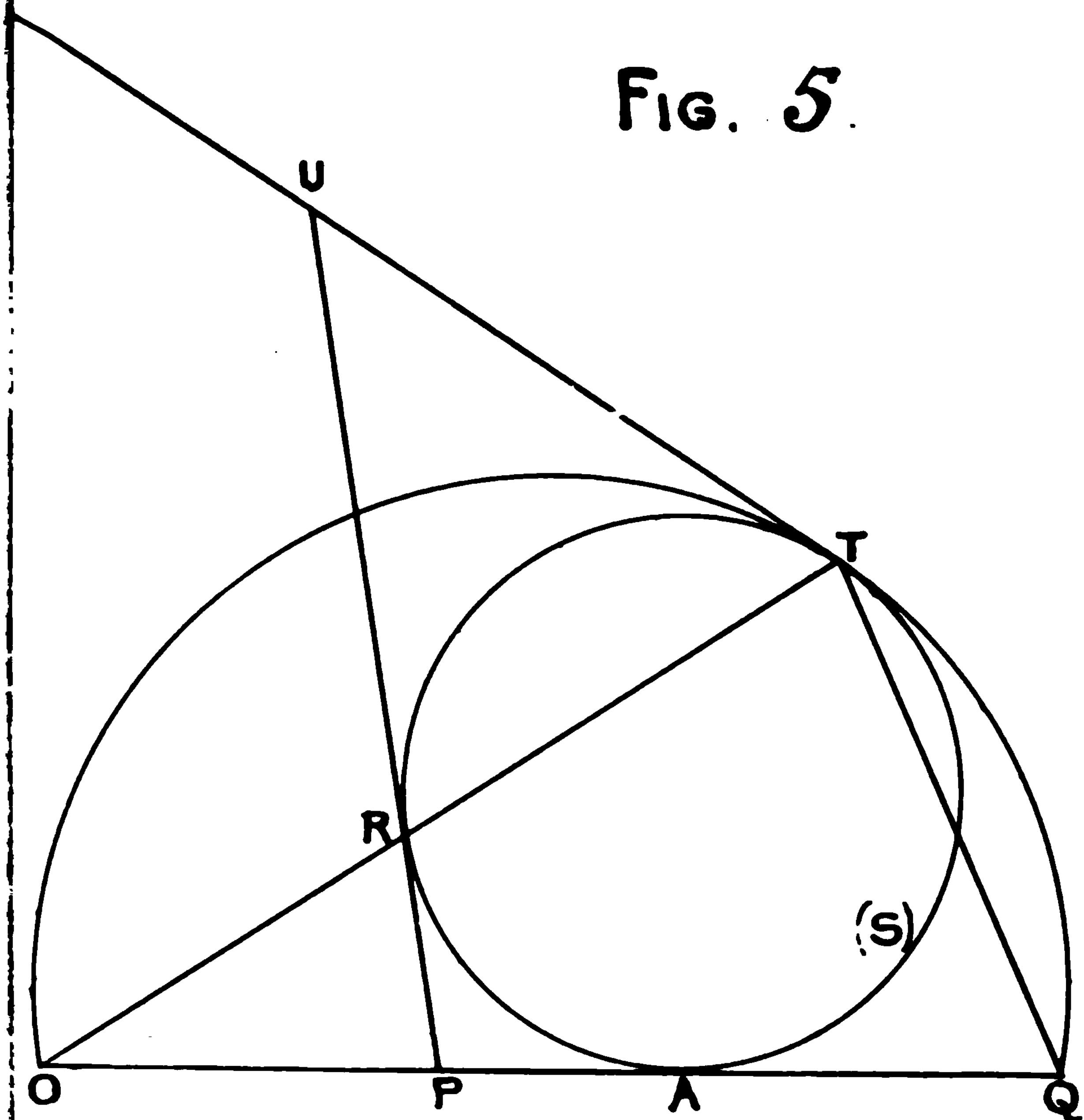


FIG. 5.



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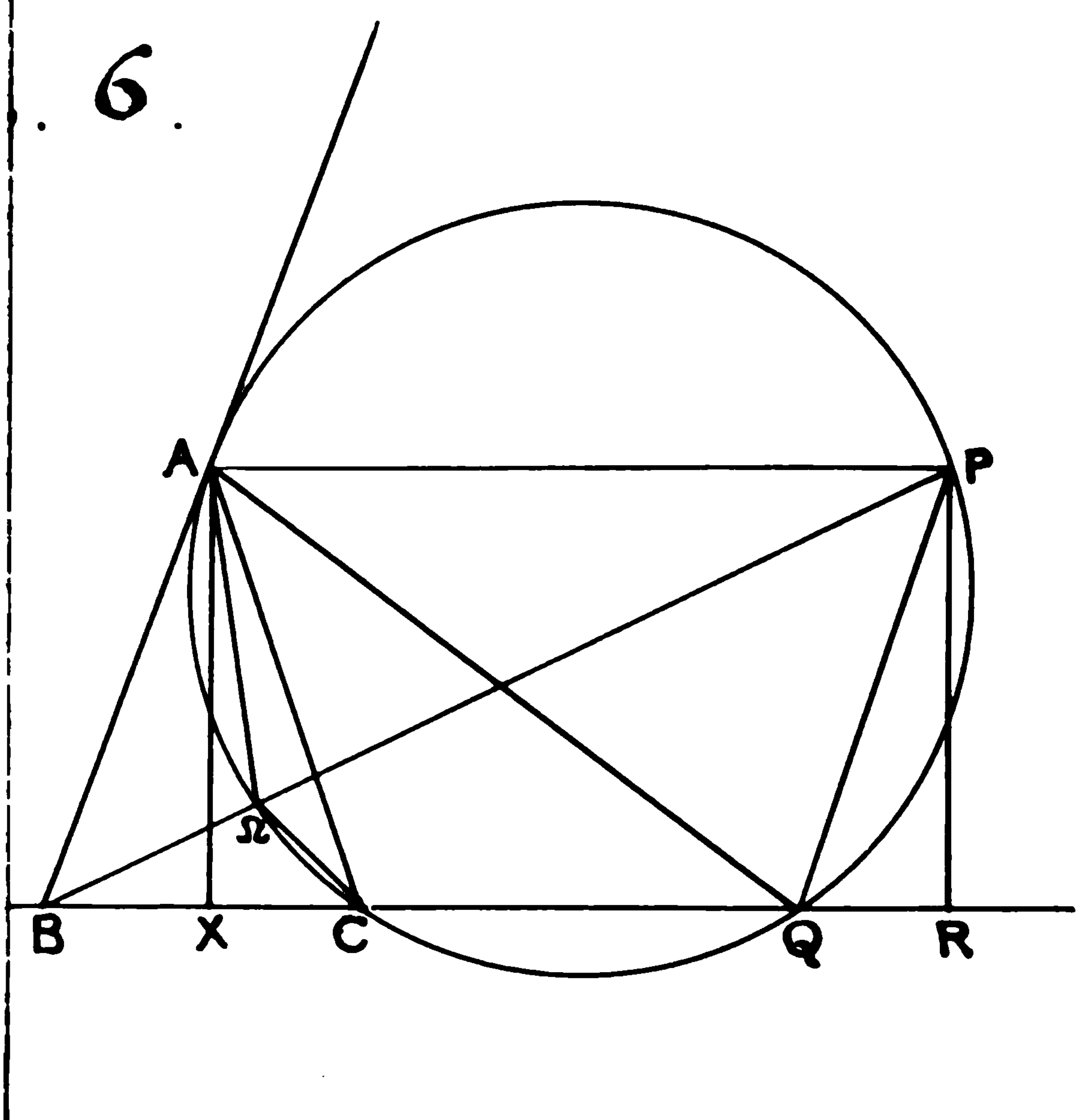
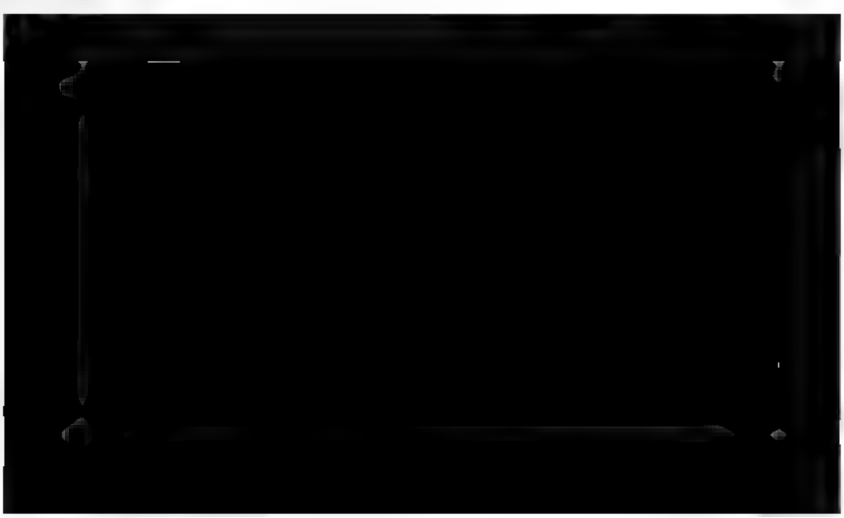
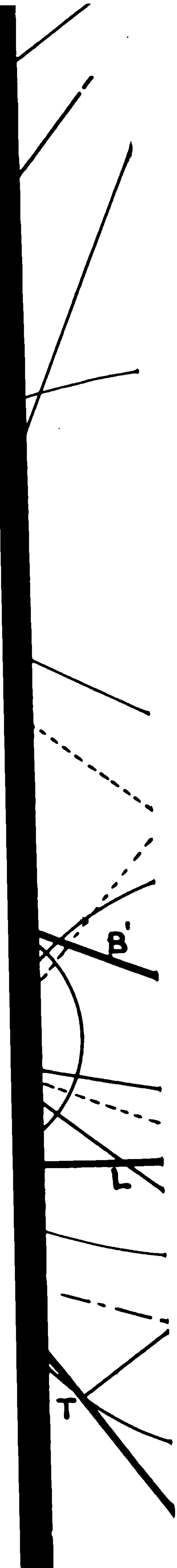
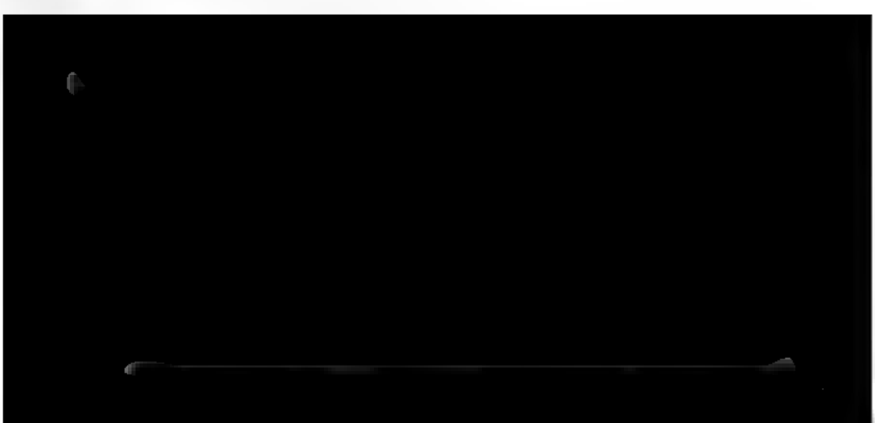






B'
L
T



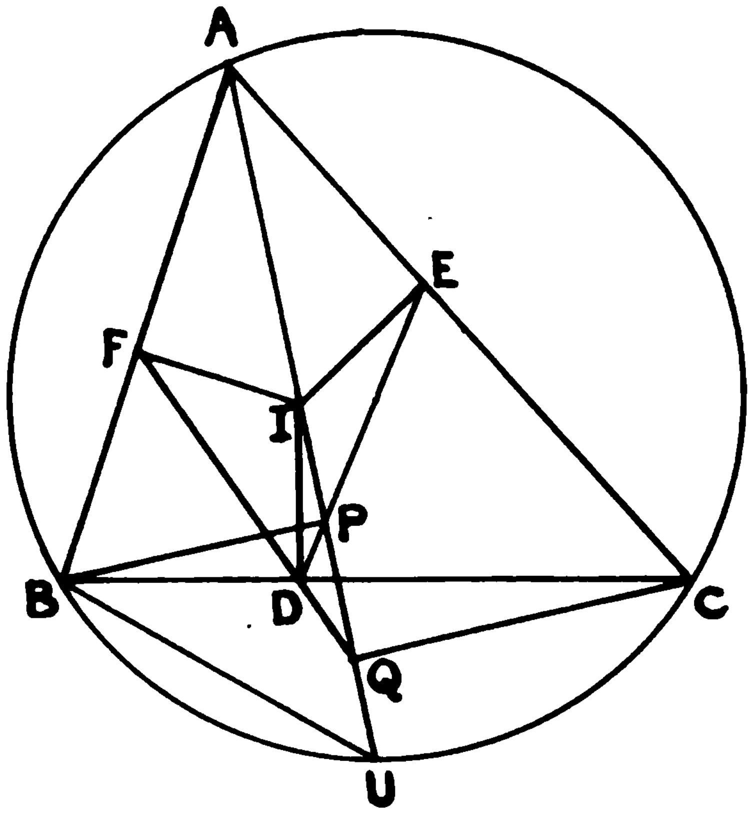


Fig. 10.

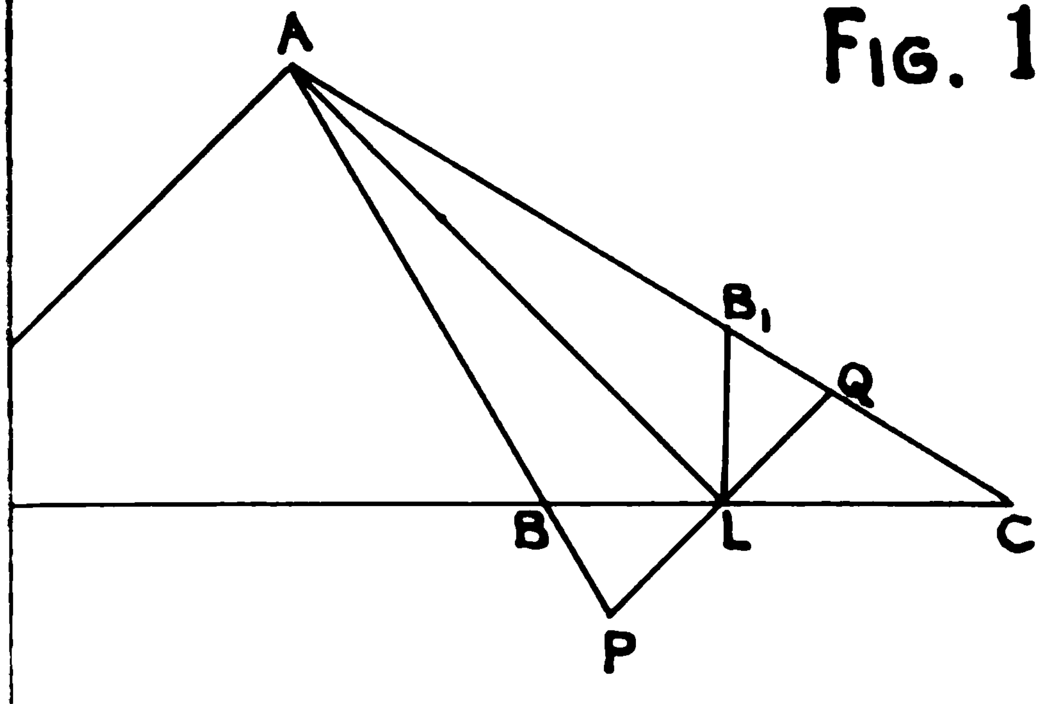


Fig. 11

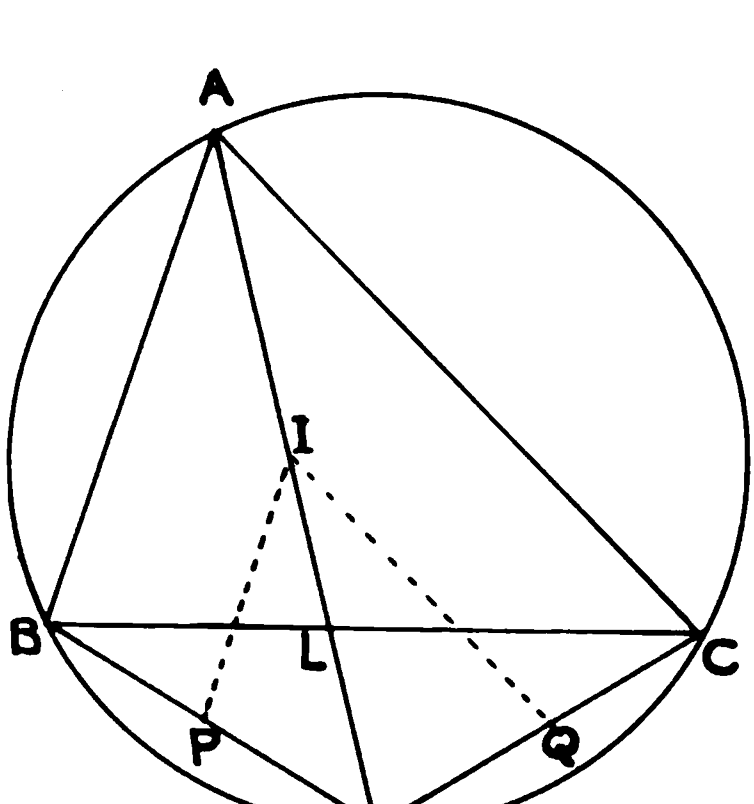


Fig. 12

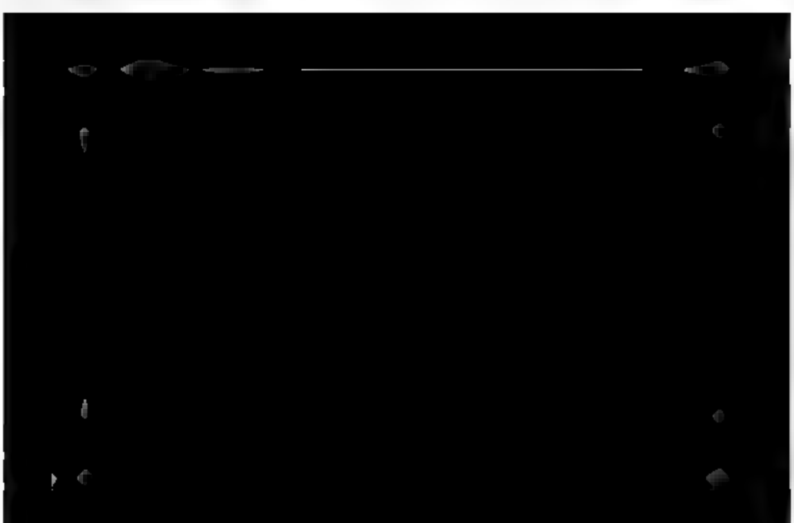


FIG. 13.

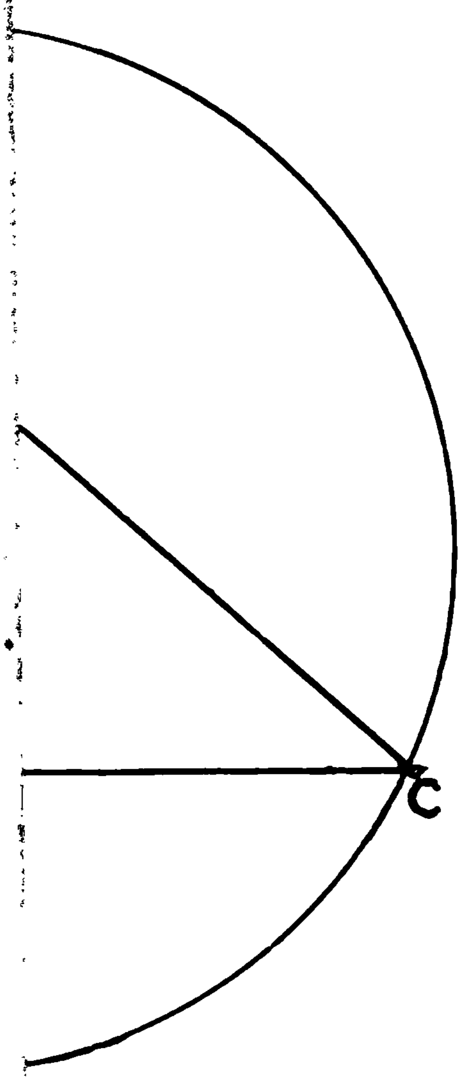


FIG 14

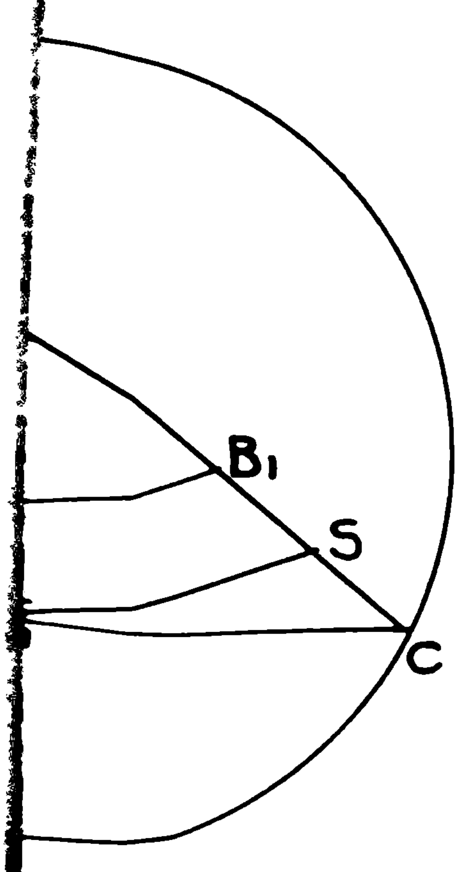


FIG. 15.

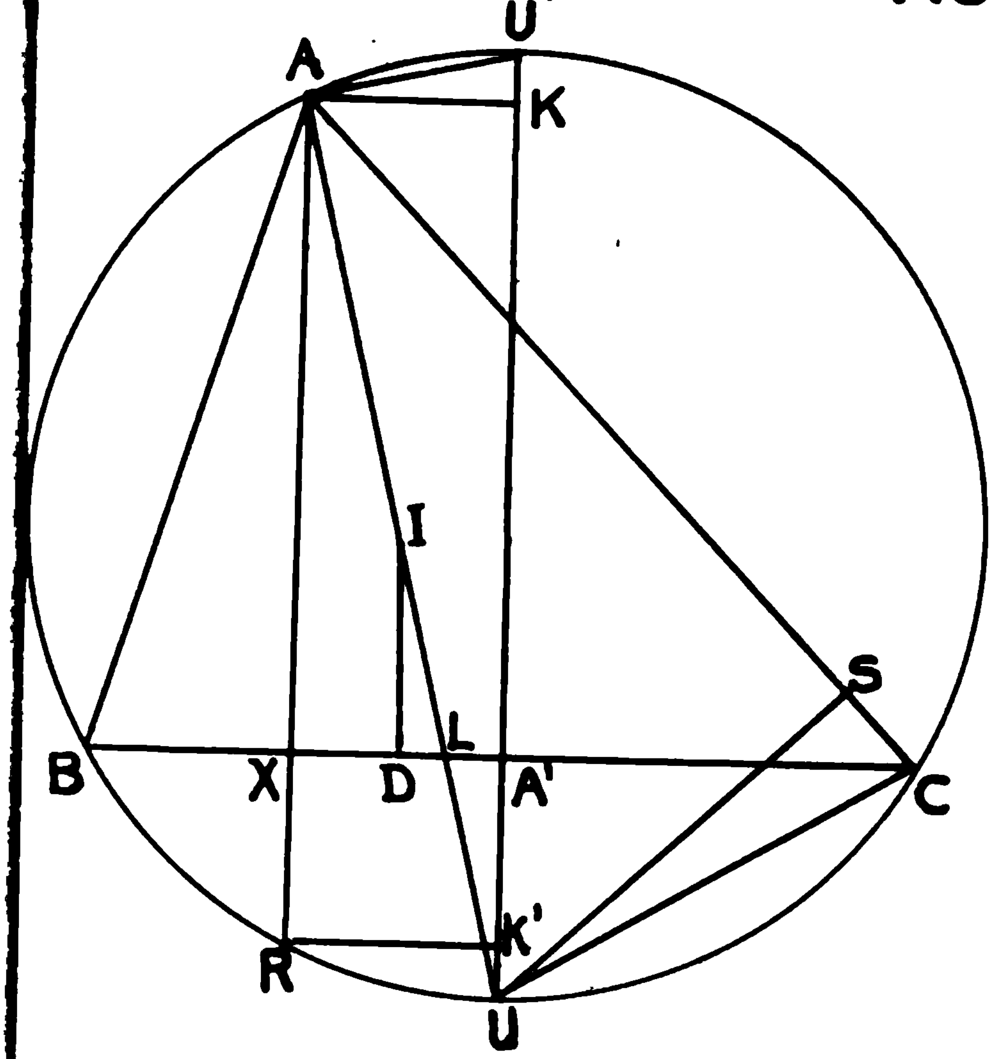
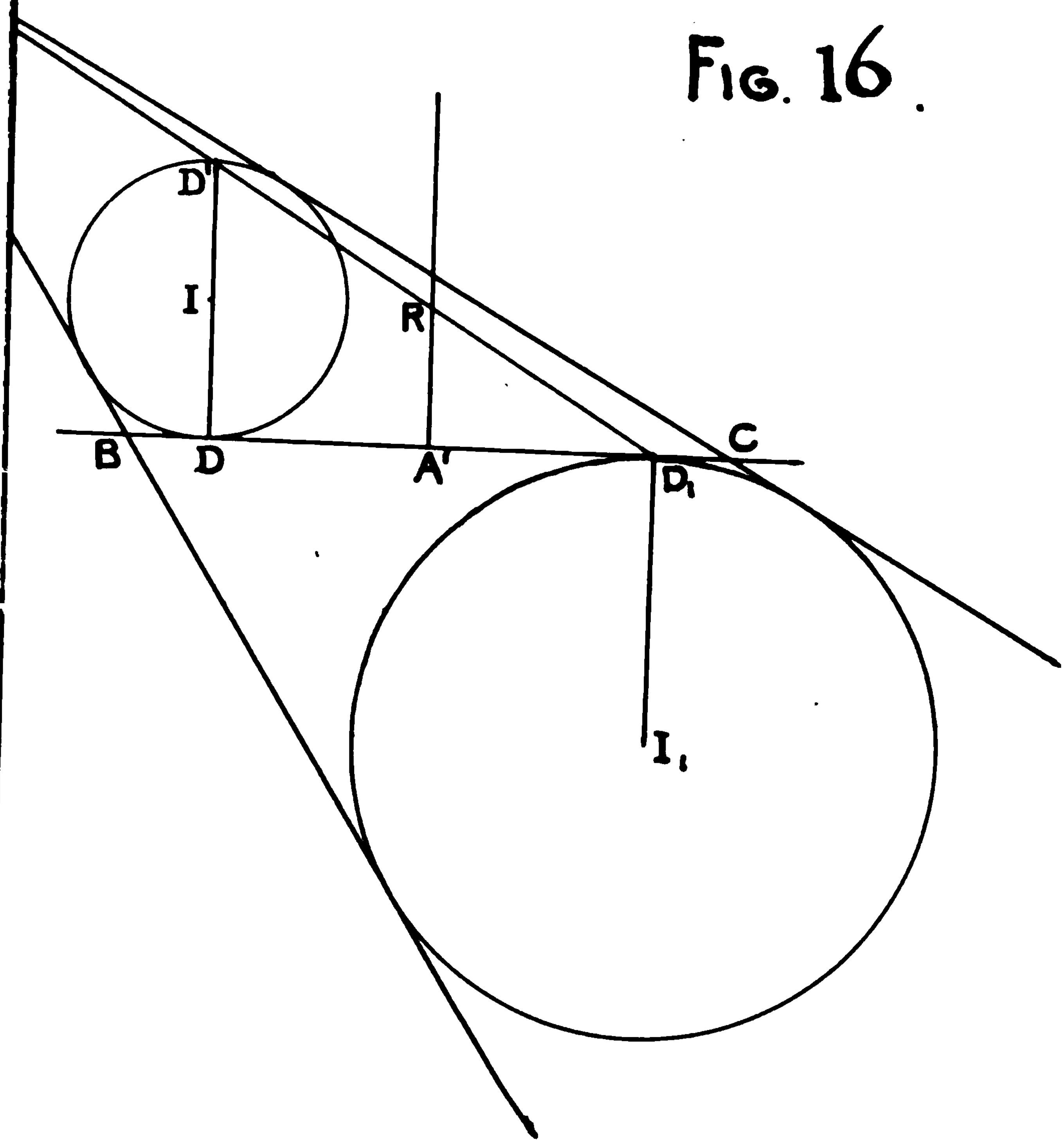
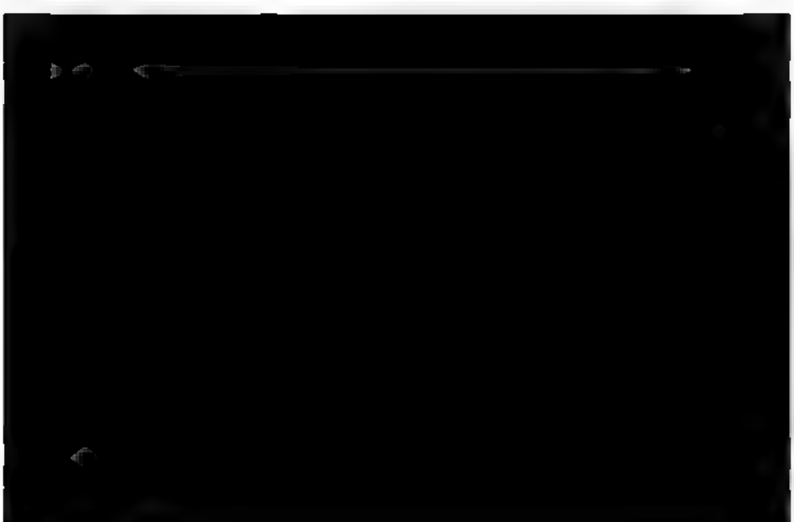
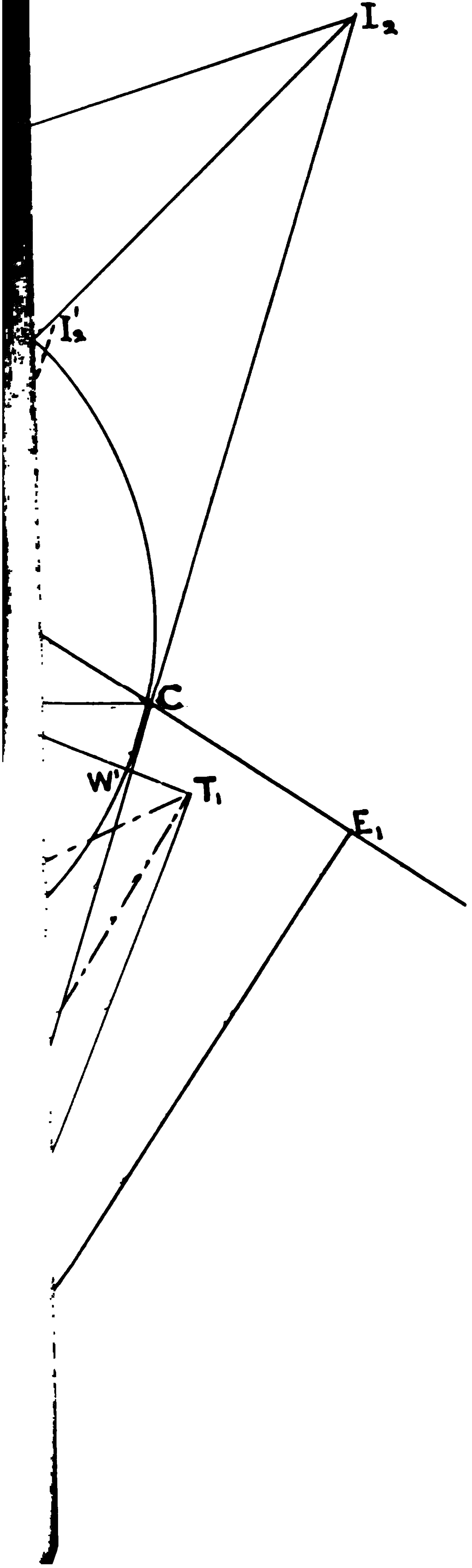
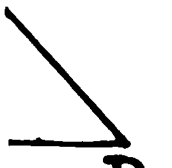


FIG. 16.






I_2
I'_2
C
W'
T_1
E_1


D_2

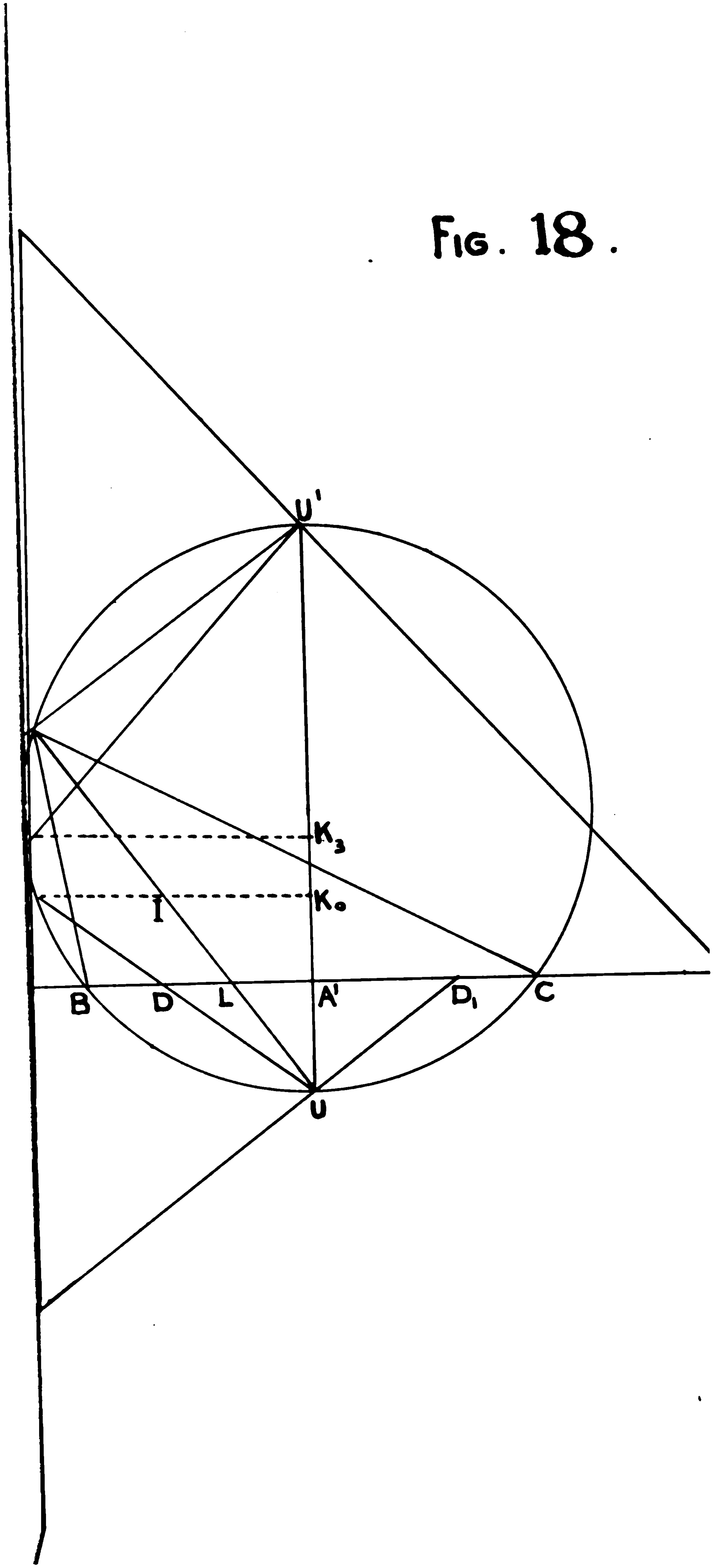
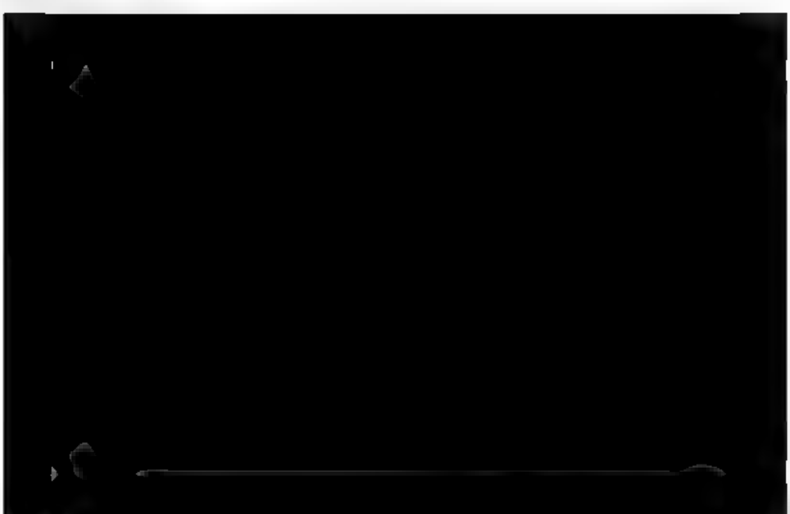
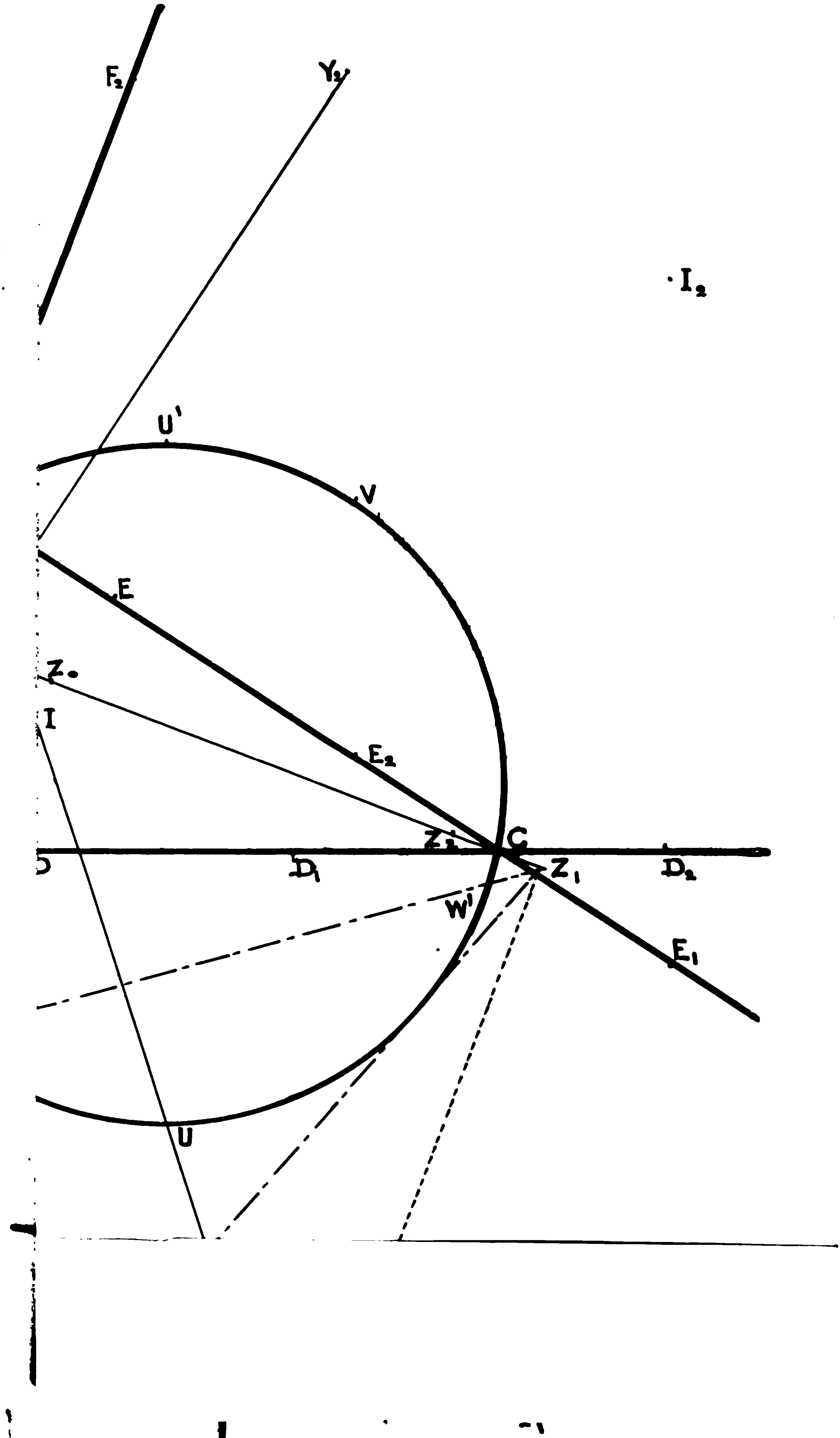


Fig. 18.





F2
Y2
I2
U'
V
E
Z0
I
E2
Z2'
C
D
D1
Z1
D2
W'
E1
U

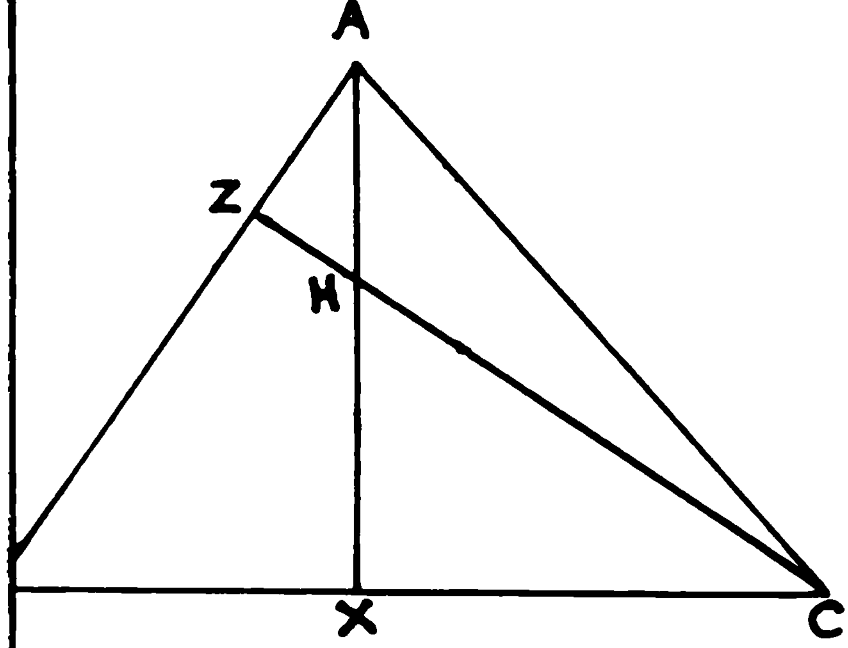
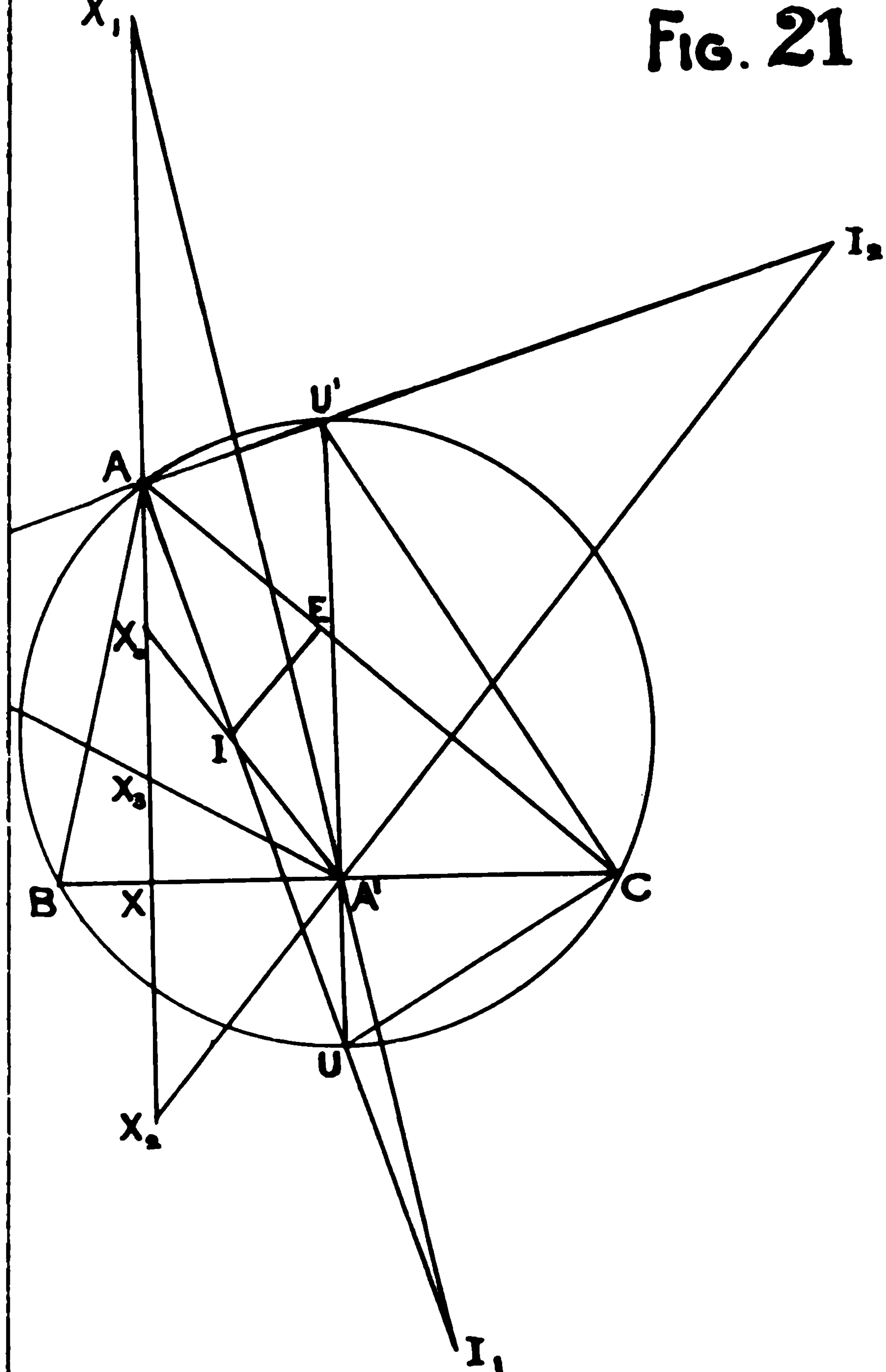


FIG. 20

FIG. 21.



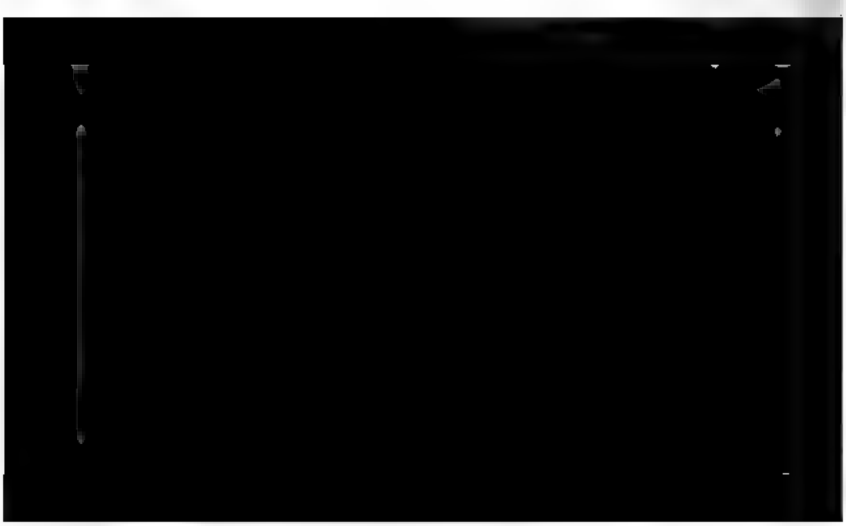
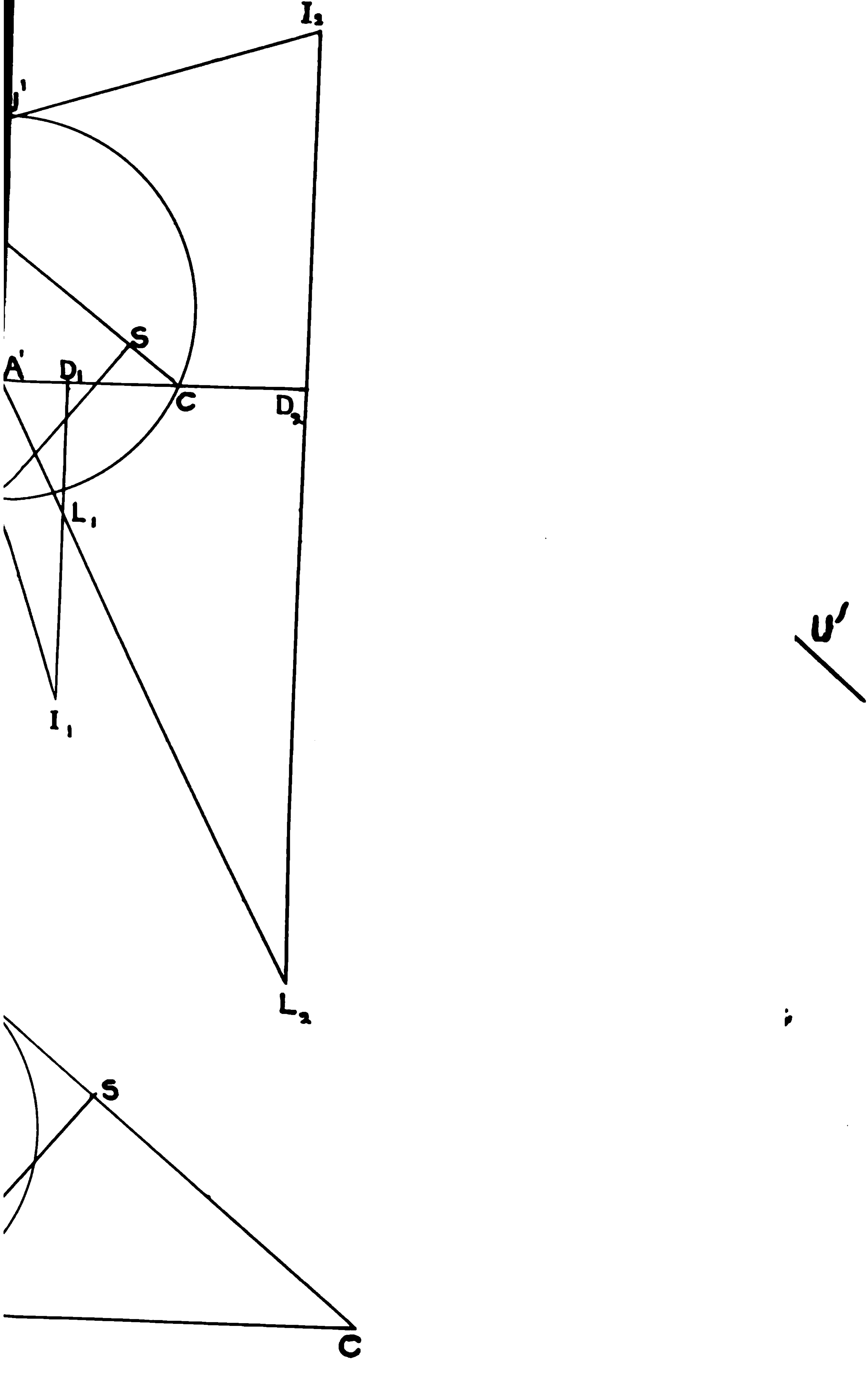
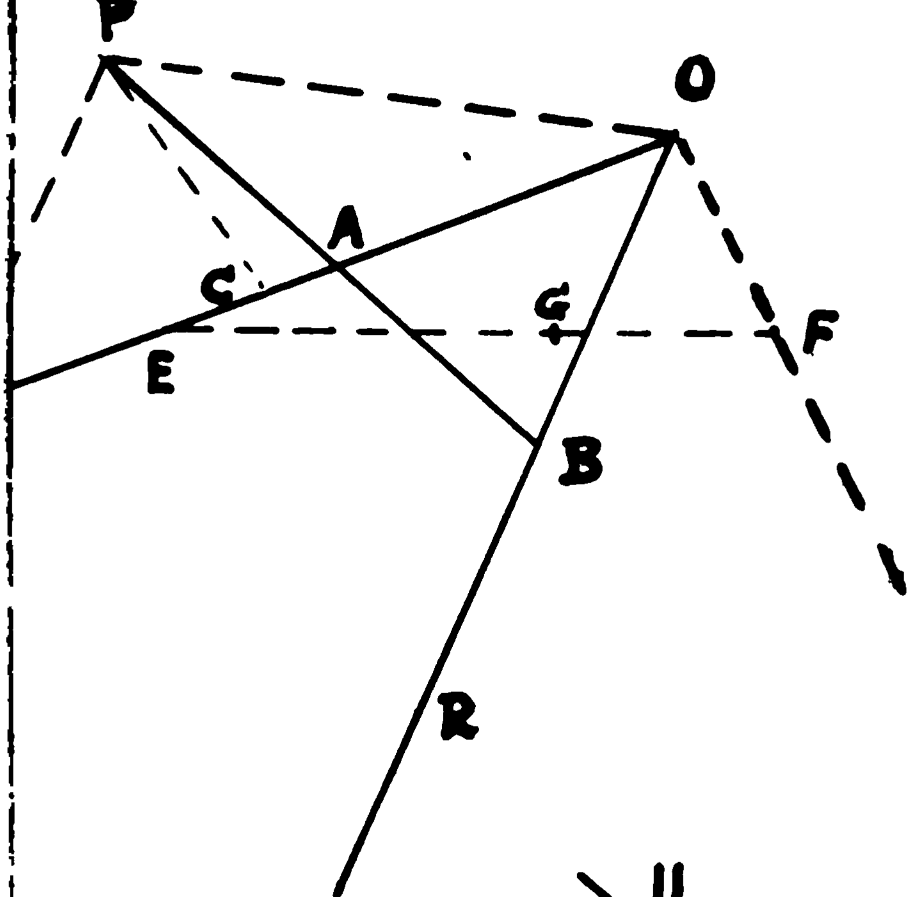
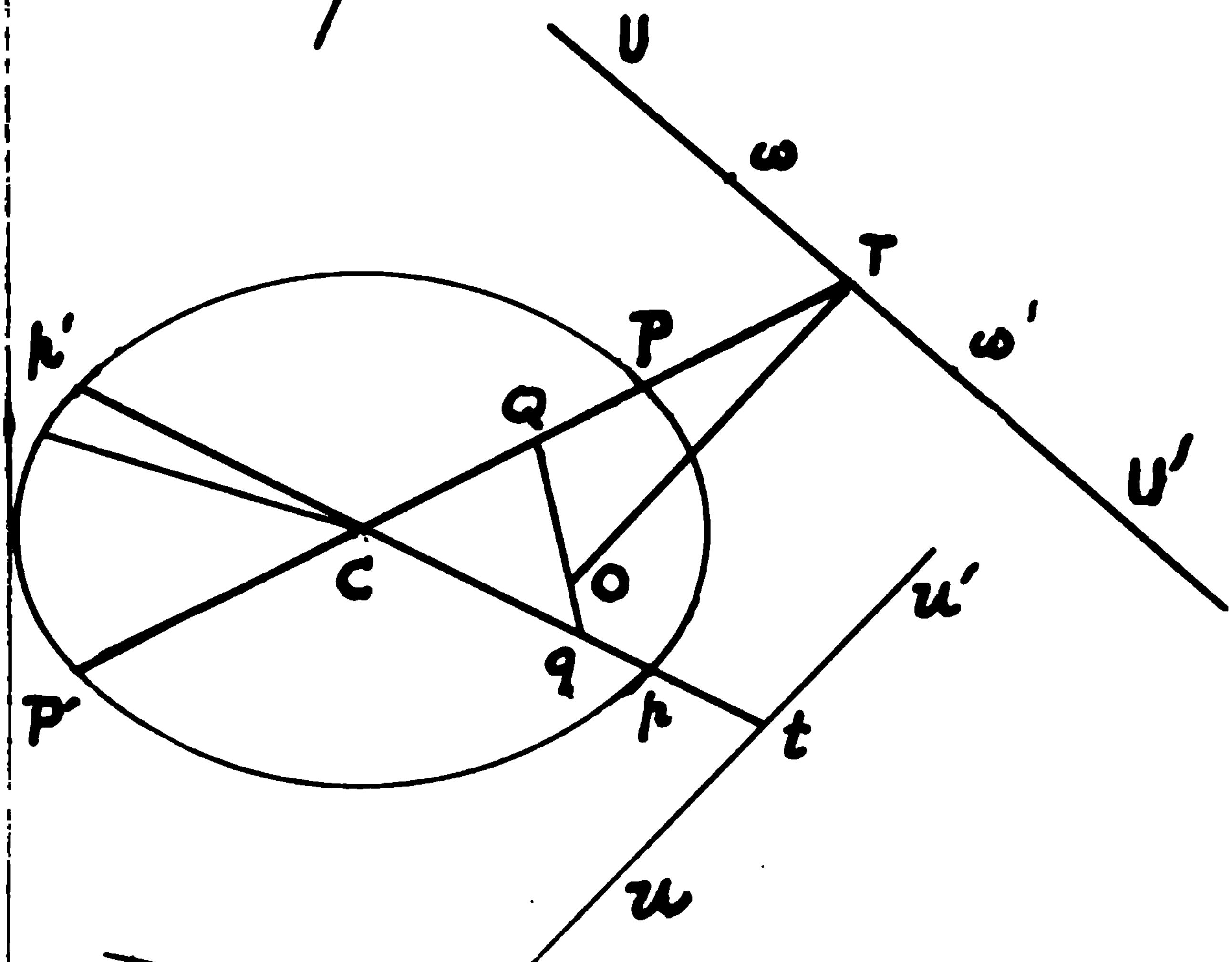
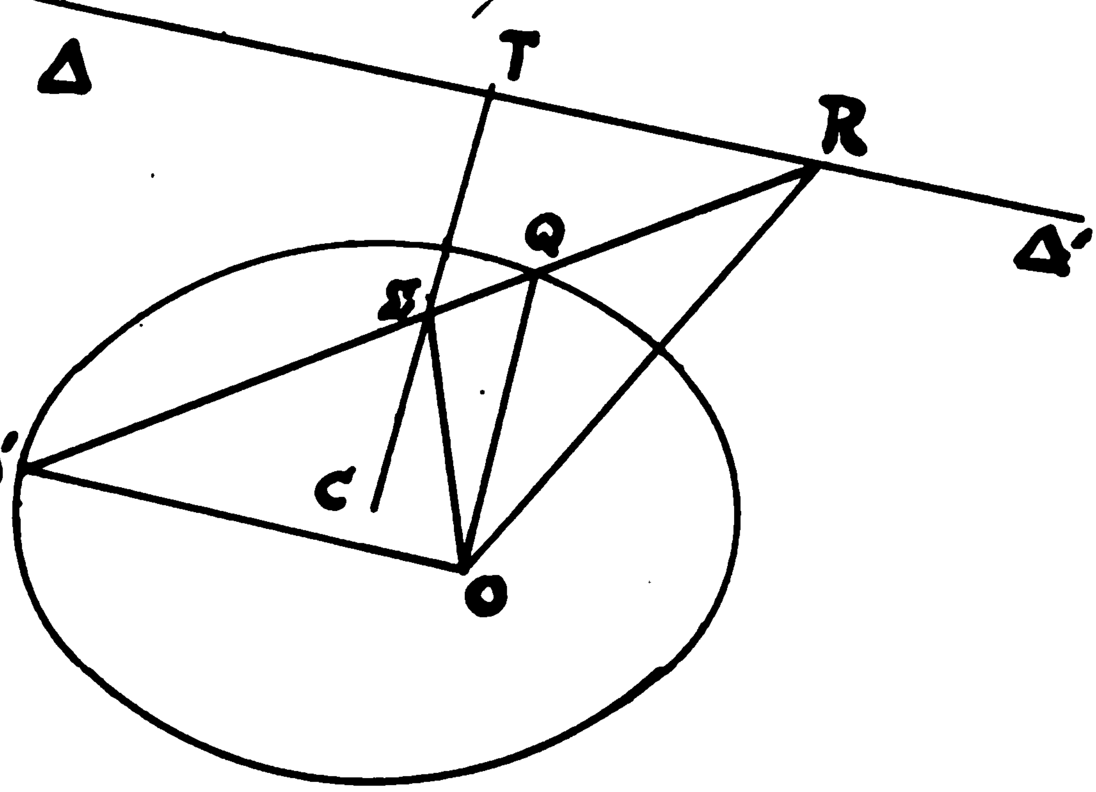


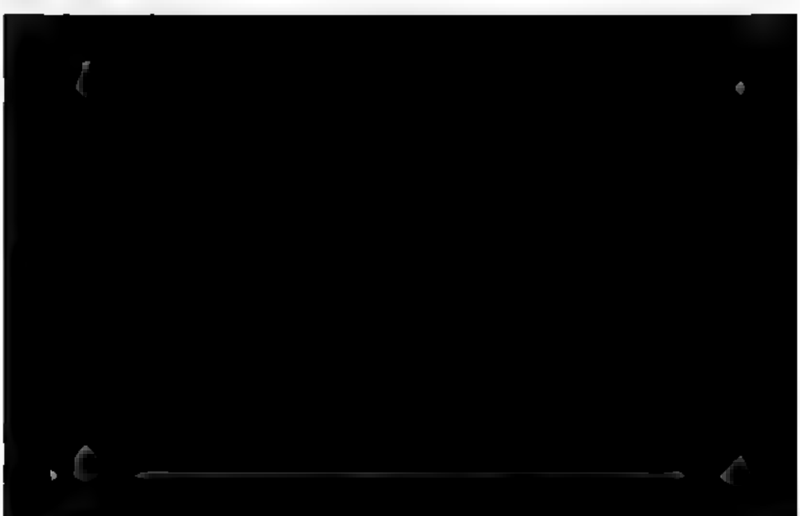
FIG. 23.

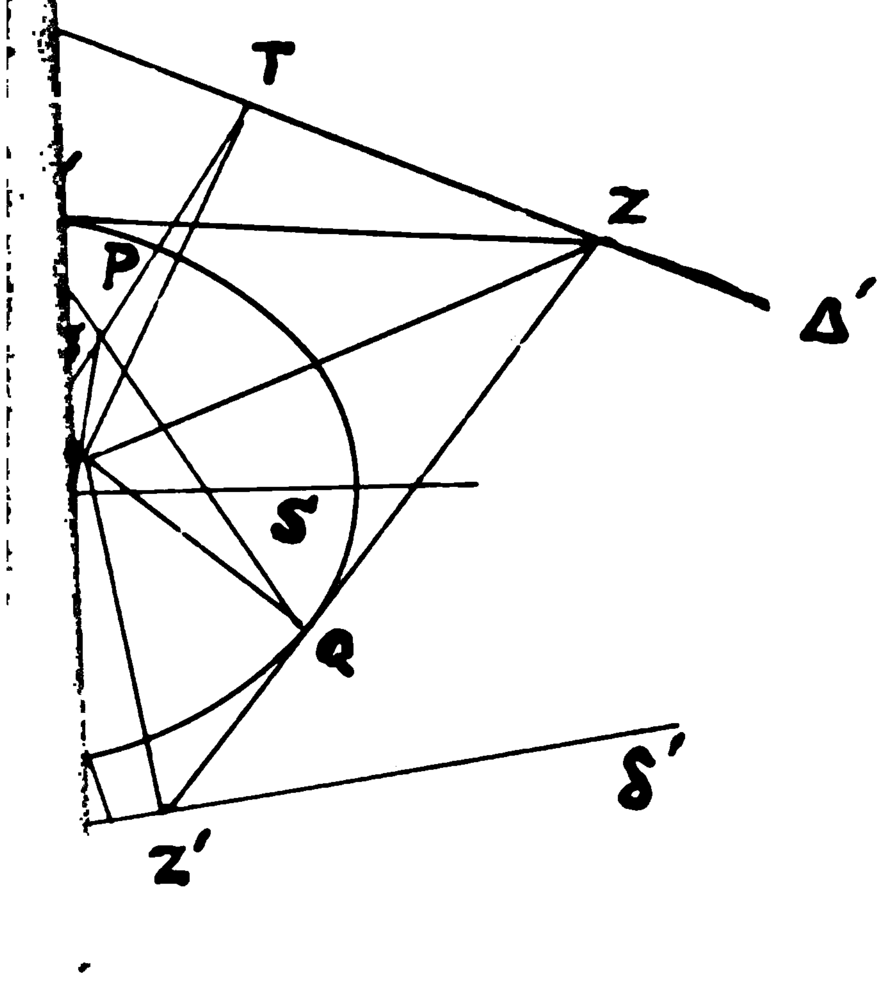



P
O
A
C
G
E
F
B
R

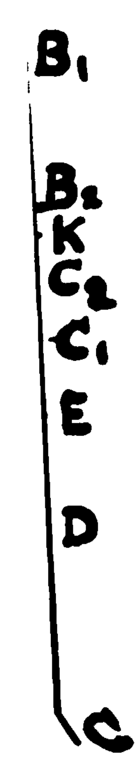

U
ω
T
ω'
U'
k'
P
Q
C
O
q
p
t
u'
u
P'

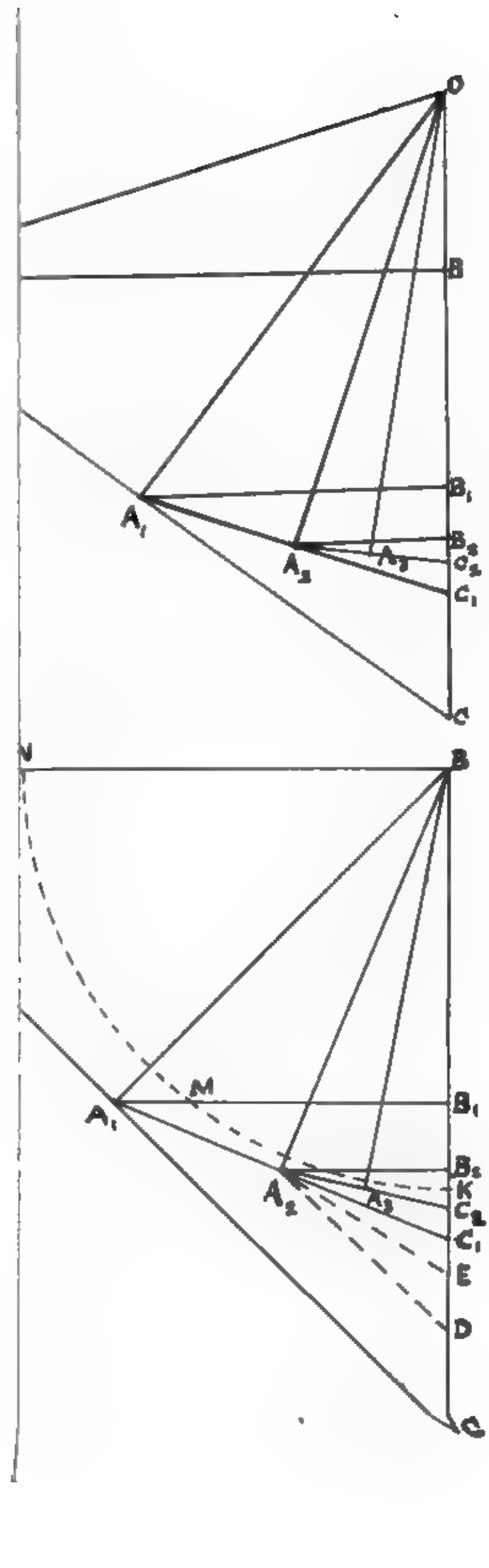

Δ
T
R
Q
Σ
Δ'
Q'
C
O




T
Z
P
Δ'
S
Q
δ'
Z'

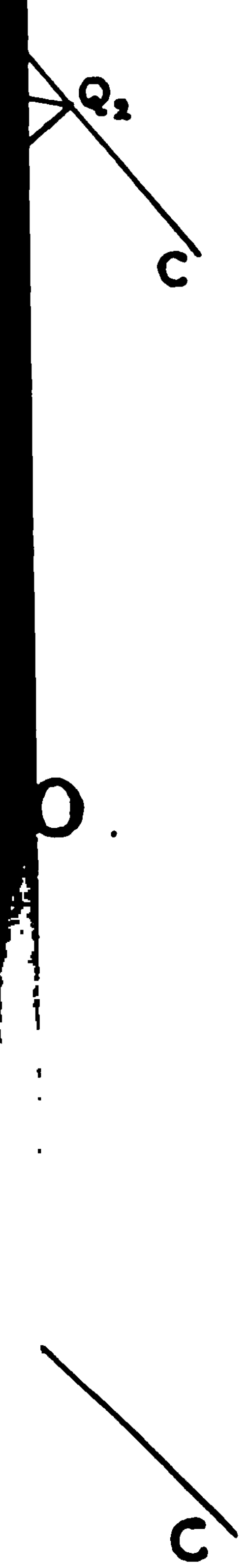

B₁
B₂
C₂
C₁
C
B


B₁
B₂
K
C₂
C₁
E
D
C



O
B
A_1
B_1
A_2
B_2
A_3
C_2
C_1
C
N
B
M
A_1
B_1
A_2
B_2
K
A_3
C_2
C_1
E
D
C

[redacted]


Q_2
C
O
C

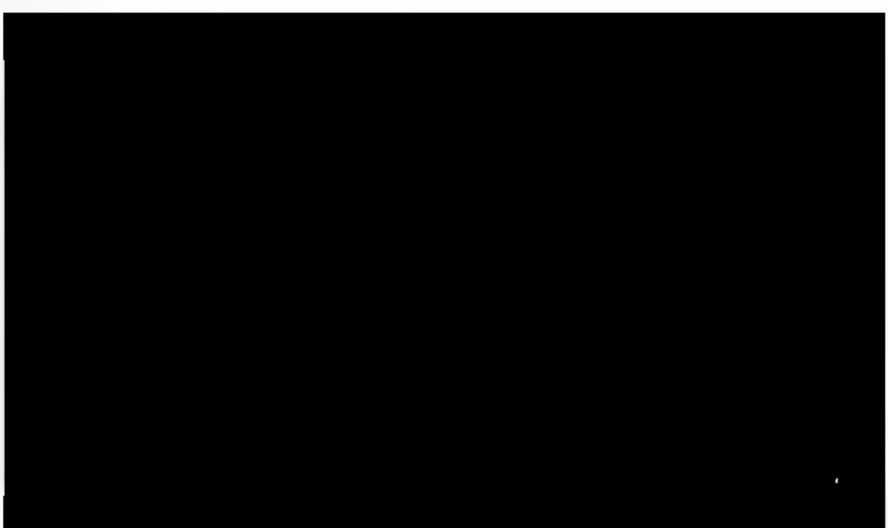
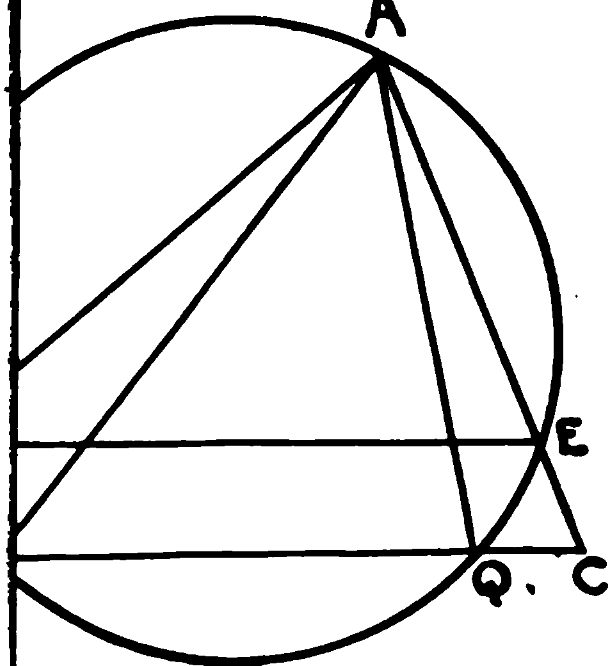


FIG. 31.



32.

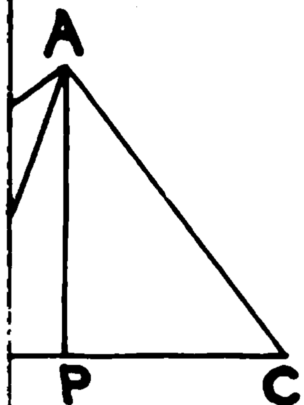
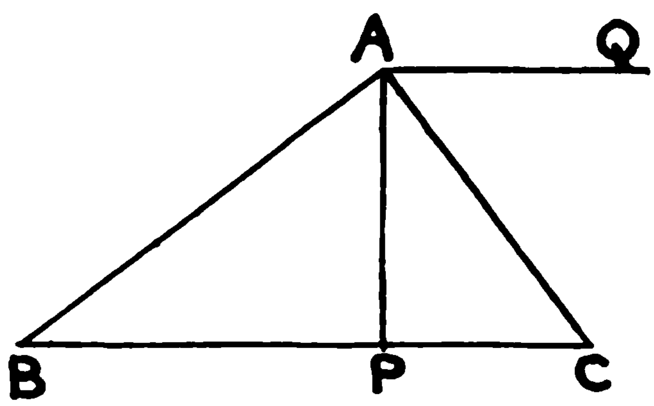


FIG. 33.



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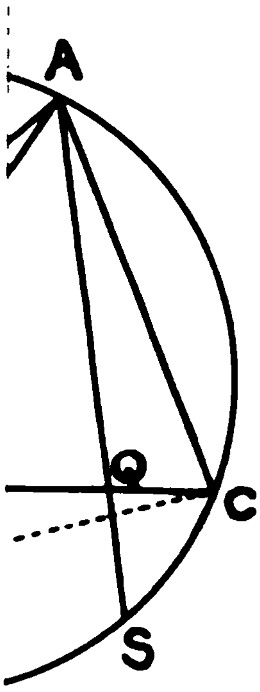
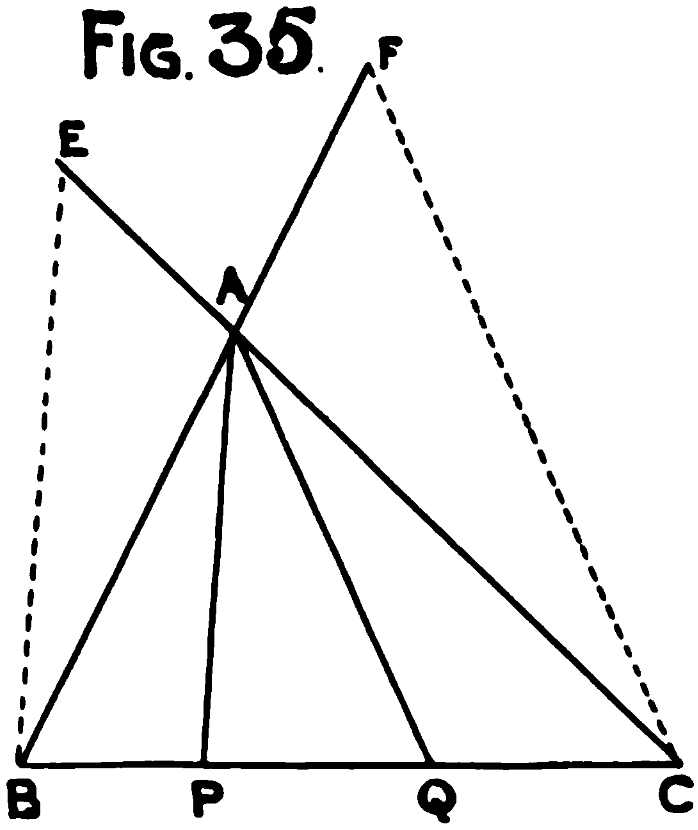


FIG. 35.



[redacted]

IG. 37.

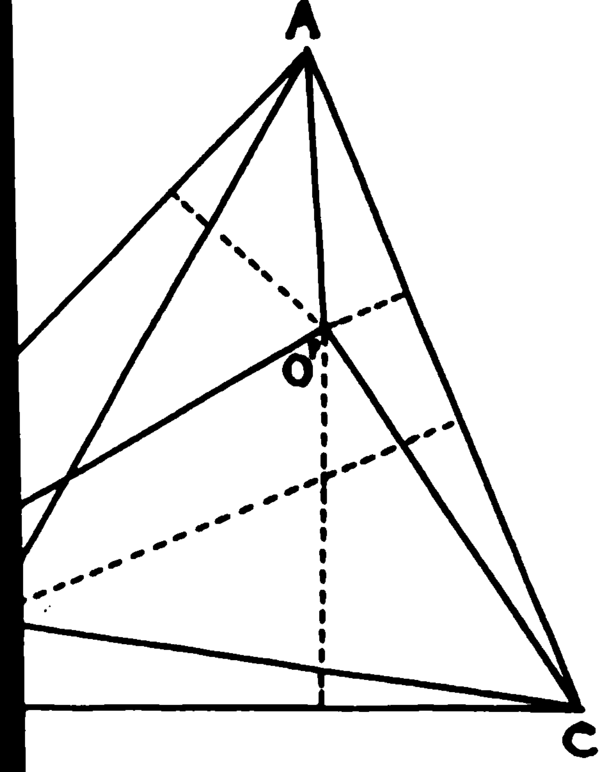
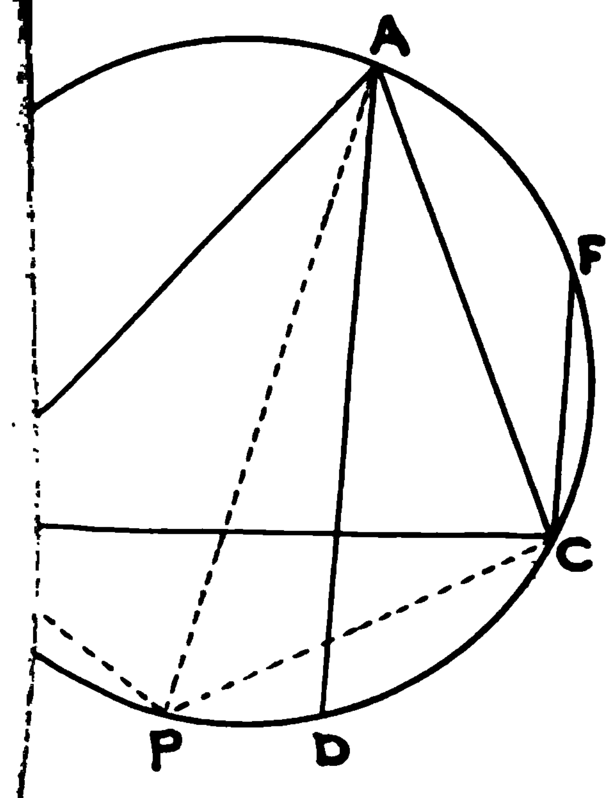


FIG. 39.



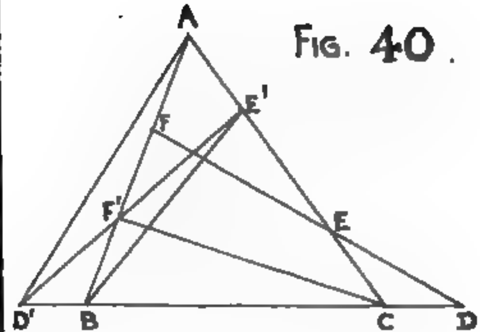


Fig. 40.

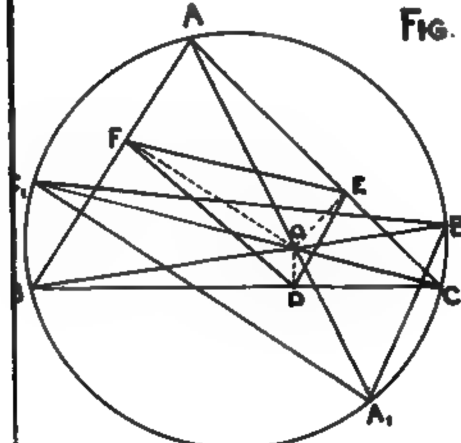


Fig. 41.

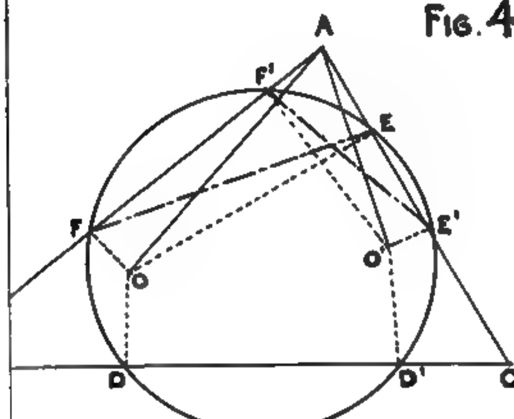
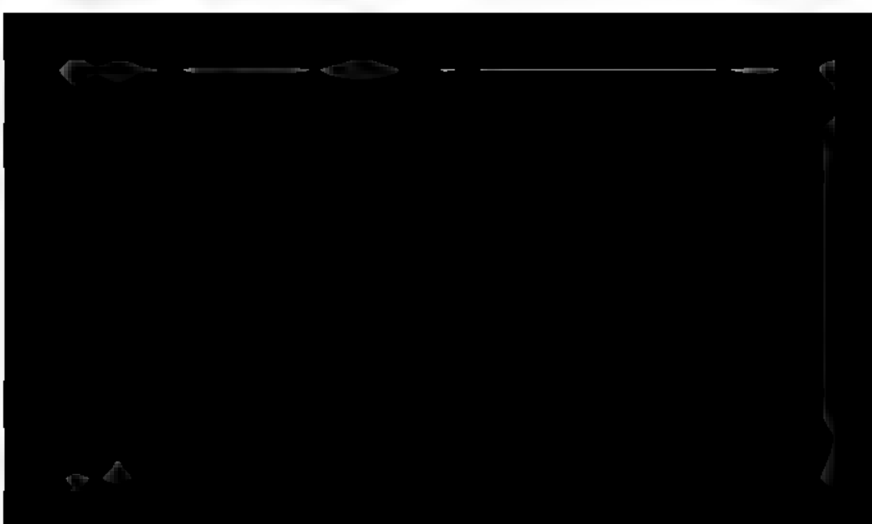


Fig. 42.



PROCEEDINGS

OF THE

EDINBURGH MATHEMATICAL SOCIETY.

VOLUME XIV.

SESSION 1895-96.

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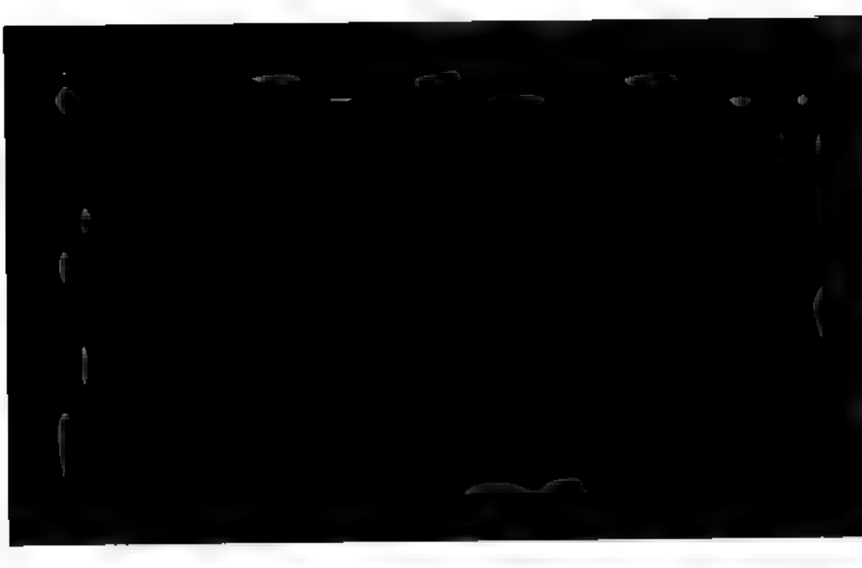

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PROCEEDINGS

OF THE

EDINBURGH MATHEMATICAL SOCIETY.

FOURTEENTH SESSION, 1895-96.

First Meeting, November 8th, 1895.

JOHN M'COWAN, Esq., M.A., D.Sc, President, in the Chair.

For this Session the following Office-bearers were elected :—

President—Mr WM. PEDDIE, D.Sc., F.R.S.E.

Vice-President—Rev. JOHN WILSON, M.A., F.R.S.E.

Hon. Secretary—Mr JOHN B. CLARK, M.A., F.R.S.E.

Hon. Treasurer—Mr DAVID TWEEDIE, M.A., B.Sc.

Editors of Proceedings { Professor C. G. KNOTT, D.Sc., F.R.S.E.
Mr W. J. MACDONALD, M.A., F.R.S.E.

Committee.

Messrs JOHN W. BUTTERS, M.A., B.Sc.; GEORGE DUTHIE, M.A.; CHARLES TWEEDIE, M.A., B.Sc.; ALEX. G. WALLACE, M.A.

[*The following paper was read 14th June 1895.*]

A Summary of the Theory of the Refraction of Thin Approximately Axial Pencils through a series of Media bounded by Coaxial Spherical Surfaces, with application to a Photographic Triplet, etc.

By Professor Chrystal.

The optical theory referred to in the title of this communication is now fully half a century old*; and has, moreover, been well expounded in the standard English treatises of Pendlebury and Heath. Still, notwithstanding its elegance and simplicity, and its great practical importance as giving the first approximation to the theory of the great majority of the optical instruments in ordinary use, its filtration into the strata of popular knowledge has been remarkably slow. It seems, therefore, to be worth while to offer a brief summary of its leading principles, freed as much as possible from the detailed calculations which become necessary when the constants of the optical system have to be deduced from the data of construction, and to indicate methods for experimental verification. In giving this summary, I shall omit the demonstrations of some of the propositions, which can be found by those who desire them in the well known *Treatise on Geometrical Optics*, by Heath.

The whole theory may be made to depend on two elementary propositions regarding the refraction of a thin pencil at a single spherical [redacted]

x' denote the distances of P and P', both measured from a point O in the axis in the same direction (which we take to be that in which the light is proceeding, always in our diagrams from left to right), then

$$Axx' + Bx + Cx' + D = 0 \quad - \quad - \quad - \quad (1),$$

*where A, B, C, D are constants depending on the radius of the refracting surface and on the indices of refraction of the media of which it is the boundary.**

The fact that the correspondence is direct, *i.e.*, that if P moves then P' always moves in the same direction, imposes a certain restriction upon A, B, C, D, viz., that BC − AD must always be negative, as we shall see presently.

The Law of Conjugate Focal Planes is accurate to a first approximation only, viz., we must suppose that the square of the distance of the point of incidence on the spherical surface from the axis is negligible in comparison with x, x', or the radius of the surface. To this degree of approximation the points corresponding to the points in any plane area (the object) perpendicular to the axis at P generate another similar plane area (the image) perpendicular to the axis at P'.

The residual phenomena which arise when we proceed to higher orders of approximation are included under the various heads of Spherical Aberration, Astigmatism, Distortion, and Curvature of the Image, with which we have nothing to do at present.

Since the object and image are similar, all that is necessary in order to determine the one when the other is given is to know the ratio of similarity, *i.e.*, the ratio of the distance between any two points in the one to the distance between the corresponding points in the other. Regarding this ratio, we have the following

Law of Helmholtz.

If β and β' be the linear dimensions of an object at P and its image at P', P and P' being axial points, and α and α' the inclinations to the axis of an incident ray through P and the corresponding emergent ray through P', then

$$\mu\beta\tan\alpha = \mu'\beta'\tan\alpha' \quad - \quad - \quad - \quad - \quad (2);$$

* See Heath, § 41-46. The theory is here stated throughout for refraction only; the case of reflection may be included by putting $\mu = -1$ for every refracting surface.

*where μ and μ' are the refractive indices of the media in the order of the passage of the rays.**

These laws can at once be extended to any number of coaxial spherical surfaces. For, if P_1' be the conjugate of P′ in the second surface, P_2' the conjugate of P_1' in the third surface, and so on, then, since each of these points is in direct projective correspondence with the one immediately preceding, the last of all, which is the image point corresponding to P after refraction by the whole system, will be in direct projective correspondence with P.

(Analytically, this involves merely the repeated application of the easily verified proposition that, if

$$\begin{aligned} A x x' + B x + C x' + D &= 0, \\ A' x' x'' + B' x' + C' x'' + D' &= 0, \end{aligned}$$

where A, B, C, D are constants, and $BC - AD$, $B'C' - A'D'$ both negative, then

$$A'' x x'' + B'' x + C'' x'' + D'' = 0,$$

where A″, B″, C″, D″ are constants depending on A, B, C, D · A′, B′, C′, D′ and such that $B''C'' - A''D''$ is negative.)

It follows therefore that, *if P and P′ be conjugate foci with respect to any system of coaxial spherical surfaces, and x and x′ denote the distances of P and P′ from any fixed point O, then*

$$Axx' + Bx + Cx' + D = 0 \quad - \quad - \quad - \quad (3),$$

[redacted]

Principal Foci, Moment, Double Points, and Stationary Points of a System.

If A do not vanish, a condition which we shall indicate when necessary by calling the system *Non-telescopic*, then the equation (3) can always be thrown into the form

$$(x-g)(x'-g') = -\gamma^2 \quad - \quad - \quad - \quad - \quad (5),$$

where $g = -C/A$, $g' = -B/A$, and $\gamma^2 = -(BC-AD)/A^2$, so that g, g', γ are all real finite constants, and γ may be taken to be either positive or negative. The constant γ^2 we shall call the *Moment of the System.*

We see at once from (5) that, when $x=\infty$, $x'=g'$, and, when $x=g$, $x'=\infty$. The two points F′ and F thus determined we call the Principal Foci of the System. The planes through F and F′ perpendicular to the axis we call the *Principal Focal Planes.* The optical property of these points is that any nearly axial pencil of parallel incident rays finally converges to or diverges from a point in a plane perpendicular to the axis through F′, and that any incident pencil converging to or diverging from a point in a plane perpendicular to the axis through F finally emerges as a parallel pencil.

When it is necessary to distinguish between these points, we may speak of F′ as the Principal Focus for incident rays, and F as the Principal Focus for emergent rays.

If we denote (see Fig. 2) the distances of the two conjugate foci P and P′ from F and F′ respectively by u and u', the former distance being reckoned positive when measured in a direction opposite to the passage of the light through the system, and the latter positive when measured in that direction, then we have the important relation

$$uu' = \gamma^2 \quad - \quad - \quad - \quad - \quad - \quad (6).$$

If p and p' denote the distances from F and F′ of any pair of conjugate foci P and P′ for incident and emergent rays respectively, and v and v' the distances of any other pair of conjugate foci Q and Q′ for incident and emergent rays from P and P′ respectively, the conventions as to sign for p and p' and for v and v' being the same as for u and u', then from (6) we deduce at once

$$p/v + p'/v' = 1 \quad - \quad - \quad - \quad - \quad (6^*).$$

Returning to the equation (5), and taking the principal focus for emergent rays, F, for origin, if we denote the distance FF′ by ∂, ∂ being positive or negative according as F′ is right or left of F (the passage of the light being supposed from left to right as usual), then (5) may be written

$$x(x' - \partial) = -\gamma^2 \quad - \quad - \quad - \quad - \quad (7).$$

Let now y denote the distance between any incident focus P and its conjugate P′, then we have from (7)

$$y = -(x^2 - \partial x + \gamma^2)/x \quad - \quad - \quad - \quad - \quad (8).$$

The points for which y vanishes, *i.e.*, the points whose conjugates are coincident with themselves, are given by

$$x^2 - \partial x + \gamma^2 = 0 \quad - \quad - \quad - \quad - \quad - \quad (9)$$

These points we may call the *Double Points of the Optical System:* the image of any object (actual or virtual) placed at a double point has the same position as the object, although it is not in general of the same size.

The double points are real, coincident, or imaginary according as $\partial^2 > = < 4\gamma^2$. *We name the optical system Hyperbolic, Parabolic, or Elliptic accordingly.*

When the double points are real they lie right or left of F according as ∂ is positive or negative.

Farther, since the sum of the distances of the double points [redacted]

In every system there are two real pairs of stationary points. Each pair is symmetrically situated with respect to F and F′, the incident points of the two pairs lie at a distance γ right and left of F respectively. In Elliptic or Parabolic systems the stationary distances are both minima. In Hyperbolic systems one of the stationary distances is a maximum the other a minimum; the incident point corresponding to the maximum always lies on the same side of F as F′.

MAGNIFICATION, PRINCIPAL POINTS, PRINCIPAL FOCAL LENGTHS, AND NODAL POINTS.

In order to deduce a rule for calculating the dimensions of the image of any given object we must combine the Law of Conjugate Focal Planes with the Law of Helmholtz.

Let Pp (Fig. 3) be a linear object of length β in the plane of the paper, P′p' its image of length β' also in the plane of the paper.

It is obvious that PF and F′P′ are incident and emergent parts of an axial ray. Let pF and Q′p' be incident and emergent parts of another ray, Q′ being a point in the Principal Focal Plane for incident rays. Since pF and PF converge to a point in the principal focal plane for emergent rays, Q′p' is parallel to F′P′. If we now join Q′P′, and take this for the emergent part of a third ray, then, since Q′ is in the principal focal plane for incident rays, the incident part of this third ray will be PQ parallel to pF.

Now, by the Law of Helmholtz,

$$\mu\beta\tan \mathrm{QPF} = \mu'\beta'\tan \mathrm{Q'P'F'},$$

that is,

$$\mu\beta\tan \mathrm{PF}p = \mu'\beta'\tan \mathrm{P'Q'}p'.$$

Hence, u and u' having the same meanings as in (6),

$$\mu\beta^2/u = \mu'\beta'^2/u'.$$

Hence

$$\frac{\beta'^2}{\beta^2} = \frac{\mu}{\mu'} \cdot \frac{u'}{u},$$

$$= \frac{\mu}{\mu'} \frac{u'^2}{\gamma^2},$$

$$= \frac{\mu}{\mu'} \frac{\gamma^2}{u^2},$$

by (6). Hence we have

$$\frac{\beta'}{\beta} = \pm\sqrt{\left(\frac{\mu}{\mu'}\cdot\frac{u'}{u}\right)} = \mp\sqrt{\left(\frac{\mu}{\mu'}\right)}\,\frac{u'}{\gamma} = \mp\sqrt{\left(\frac{\mu}{\mu'}\right)}\cdot\frac{\gamma}{u} \quad - \quad (10),$$

the upper or lower signs to be taken according as the image is erect or inverted or *vice versa*. By means of this relation the magnification of an object placed at any given point may be calculated.

There are two pairs of conjugate foci for which the magnification is unity corresponding to

$$u = \sqrt{(\mu/\mu')}.\gamma, \qquad u' = \sqrt{(\mu'/\mu)}.\gamma \quad - \quad - \quad (11);$$

$$u = -\sqrt{(\mu/\mu')}.\gamma, \qquad u' = -\sqrt{(\mu'/\mu)}.\gamma \quad - \quad - \quad (12).$$

For one of these pairs the image is equal to the object and erect; for the other equal and inverted. Which is which depends on the absolute sign of γ, a quantity whose square alone has hitherto been defined; so that we cannot distinguish between the two pairs without farther examination of the special system. We know, however, that there are always two real pairs of the kind described. The pair for which the image is erect are called the *principal points* of the system (H, H′); the other pair we may call the *anti-principal points* (K, K′).

The distances of the principal points of incidence and emergence from the principal foci of incidence and emergence respectively are called the principal focal lengths of the system. If we denote these

[redacted]

It will be observed that in the general case which we are now considering the principal points coincide neither with the stationary points nor with the double points. The magnification for the former is

$$\pm \sqrt{(\mu/\mu')};$$

for the latter $\pm \sqrt{(\mu/\mu')}\{\partial/2\gamma \pm \sqrt{((\partial/2\gamma)^2 - 1)}\}$.

The line joining the points where any incident ray and its corresponding emergent ray meet the principal planes of incidence and emergence respectively is parallel to the axis.

A similar proposition holds for the anti-principal planes, with the variation that the joining line passes through the point bisecting KK′.

To prove the first of these propositions we have only to remark that, since the principal planes are conjugate, the points Q and Q′ (Fig. 4) in which any ray meets them are conjugate foci, and so also are the principal points H, H′ themselves. Hence H′Q′ is the image of HQ; and therefore by the fundamental property of the principal points H′Q′ = HQ, both being like directed. Hence QQ′ is parallel to HH′. A similar proof establishes the second proposition.

By means of the principal planes we can readily construct in a variety of ways the emergent ray corresponding to any given incident ray.

Let the incident ray meet the principal focal plane of emergence in R (Fig. 5), and the principal plane of incidence in S. Draw SS′ and RT′ parallel to the axis meeting the principal plane of emergence in S′ and T′; then S′P′ parallel to T′F′ is the emergent ray corresponding to PS, as is at once obvious if we notice that RS, RT form a pencil diverging from a point in the principal focal plane of emergence. P′ is of course the conjugate of P.

We can also construct the conjugate of any point P not lying on the axis as follows:—

Let PR (Fig. 6) parallel to the axis meet the principal plane of emergence in R′, and PF meet the principal plane of incidence in S. Then the parallel to the axis through S meets R′F′ in P′ the conjugate of P.

Similar constructions can be effected by means of the anti-principal planes, if we replace the parallels to the axis by lines drawn through the middle point of FF′.

By means of Helmholtz's Law and the relations of (13) and (16) we find

$$\frac{\tan\alpha'}{\tan\alpha} = \mp\frac{u}{f'} \qquad - \quad - \quad - \quad - \quad (18),$$

where α and α' are the inclinations to the axis of any incident ray passing through P and the corresponding emergent ray passing through the conjugate focus P'.

We see from (18) that *in every optical system there are two real pairs of axial conjugates, given by*

$$\begin{aligned} u &= -f', \; u' = -f, \\ u &= f', \; u' = f \end{aligned} \qquad - \quad - \quad - \quad - \quad (19).$$

which have the property that the corresponding incident and emergent rays passing through them make equal angles with the axis. For one of these pairs, which are called the *nodal points*, the incident and emergent rays are parallel : for the other pair, which may be called the *antinodal points*, the incident and emergent rays are equally but oppositely inclined to the axis.

By considering a figure in any special case it is easy to see that if N and N' (Fig. 7) be the nodal points, H and H' the principal points, then N and H on the one hand and N' and H' on the other must always lie on the same side of F and F' respectively.

From (16) we see that the magnification corresponding to the nodal points is $+f/f' = u/u'$.

[redacted]

Since $\mu = \mu'$, we have from (13) $f = f' = \gamma$ - - - (13′).

Hence *the stationary points and the nodal points coincide with the principal points.* It is therefore sufficient in any such system to confine our attention merely to the four points F, F′, H, H′. Since these are now symmetrically disposed about the middle point of FF′, we may speak of this point as the *centre of the system* and call the system *symmetrical.*

The fundamental formulæ (4), (15), (16), (17), (18) now become

$$\beta \tan\alpha = \beta' \tan\alpha' \quad - \quad - \quad - \quad (4');$$

$$uu' = f^2 \quad - \quad - \quad - \quad (15');$$

$$\frac{\beta'}{\beta} = \pm\sqrt{\left(\frac{u'}{u}\right)} = \mp\frac{u'}{f} = \mp\frac{f}{u} \quad - \quad - \quad (16');$$

$$\frac{1}{v} + \frac{1}{v'} = \frac{1}{f} \quad - \quad - \quad - \quad - \quad (17');$$

$$\frac{\tan\alpha'}{\tan\alpha} = \mp\frac{u}{f} \quad - \quad - \quad - \quad - \quad (18').$$

Classification of Symmetrical Optical Systems.

Since symmetrical systems are of great importance, it seems to be worth while to classify the fundamentally distinct kinds that can arise and to indicate how typical models of them can be constructed for the purposes of laboratory instruction.

We may suppose one of the four determining points F, F′, H, H′ kept fixed. There arise therefore as many distinct cases as there are distinct topological arrangements of the three remaining points relative to F and to each other. Bearing in mind the symmetry of the system, we thus get the following eight distinct cases :—

F H H′F′, F H′H F′, H′F F′H, H′F′F H ;
F′H′H F, F′H H′F, H F′F H′, H F F′H′.

To remove the latent ambiguity arising from the indeterminate sign of γ in the general theory we divide the eight systems into two classes, viz., *Inverting Systems*, which give an inverted image of a distant object; and *Erecting Systems*, which give an erect image of a distant object. By considering the construction given above for the image of a non-axial point, it is easily seen that a system is Inverting or Erecting according as the direction of FH is the same as or opposite to the direction of the passage of the light through the

system. If, therefore, we suppose, as hitherto, the light to pass from left to right, we see at once that the first four systems above set down are inverting and the rest erecting.

The following table indicates farther the nature of the systems as to the reality of the double points and the nature of the stationary distances HH′ between the principal points and KK′ between the anti-principal points.

Inverting Systems.	Erecting Systems.	Nature as to Double points.	H H′	K K′
F H H′F′	F′H′H F	Hyperbolic	Maximum	Minimum
F H′H F′	F′H H′F	Elliptic	Minimum	Minimum
H′F F′H	H F′F H′	Elliptic	Minimum	Minimum
H′F′F H	H F F′H′	Elliptic or Hyperbolic	Minimum	Minimum or Maximum

There are of course transition cases such for example as a system for which $FF' = 0$, $FH' = 0$, $FH = 0$, or $HH' = 0$. The last corres-
[redacted]

between the lenses being c (positive), then by specialising the formulæ given by Heath, § 60, we find

$$\left.\begin{aligned}
\mathrm{HH}' &= -c^2/(f+f'-c)\,; \\
\mathrm{FF}' &= (2ff'-c^2)/(f+f'-c)\,; \\
\mathrm{FH} &= (ff')/(f+f'-c)\,; \\
\mathrm{FH}' &= (ff'-c^2)/(f+f'-c). \\
\mathrm{KK}' &= (4ff'-c^2)/(f+f'-c)
\end{aligned}\right\} \qquad (20)$$

The following table, therefore, gives the conditions of construction for our eight types :—

Inverting	1	F H H'F'	$c>f+f'$	$ff'<0$	
	2	F H'H F'	$c<f+f'$	$ff'>0$	$ff'>c^2$
	3	H'F F'H	$c<f+f'$	$ff'>0$	$c^2>ff'>\frac{1}{2}c^2$
	4	H'F'F H	$c<f+f'$	$ff'>0$	$ff'<\frac{1}{2}c^2$
Erecting	5	F'H'H F	$c<f+f'$	$ff'<0$	
	6	F'H H'F	$c>f+f'$	$ff'>0$	$ff'>c^2$
	7	H F'F H'	$c>f+f'$	$ff'>0$	$c^2>ff'>\frac{1}{2}c^2$
	8	H F F'H'	$c>f+f'$	$ff'>0$	$ff'<\frac{1}{2}c^2$

(21.)

The only systems for which lenses of negative focal length are absolutely required are (1) and (5). All the others, except (7), can be constructed with two identical lenses of equal positive focal length placed at the proper distance apart. Perhaps the simplest way to work out a complete set of models is to take advantage of the fact that (5) can be derived from (1) merely by altering c; and that (2), (3), (4) can be converted into (6), (7), (8) respectively by changing the signs of f and f'.

The following table gives a convenient set of models. As it is convenient to know the positions of the anti-principal points when the combinations have to be measured experimentally, I have indicated them by the letters KK' in the table :—

[redacted]

[redacted]	H H′	F F′	F H	F H′	K K′
[redacted]	+ 3·6f	+ 13·6f	+ 5f	+ 8·6f	+ 23·6f
[redacted]	− ·26f	+ 1·17f	+ ·71f	+ ·46f	+ 2·6f
[redacted]	− 1·80f	+ ·70f	+ 1·25f	− ·55f	+ 3·20f
[redacted]	− 9·63f	− 2·96f	+ 3·33f	0·30f	+ 3·70f
[redacted]	− 4·5f	9·5f	− 2 5f	− 7f	− 14·5f
[redacted]	+ ·14f	− ·63f	− ·38f	− ·25f	− 1·38f
[redacted]	+ ·45f	− ·17f	− ·31f	+ ·14f	− ·80f
[redacted]	+ ·78f	+ ·24f	− ·27f	+ ·51f	− ·30f

(22).

With six lenses which can be bought for a few shillings, and a couple of draw-tubes for adjusting them at different distances apart, apparatus can be constructed by means of this table for illustrating the various typical systems and for exercising students in measuring their constants.

TELESCOPIC SYSTEMS.

Hitherto we have supposed that the constant A in the equation

$$Axx' + Bx + Cx' + D = 0 \qquad - \quad - \quad - \quad (3)$$

does not vanish. We shall now briefly consider systems for which $A = 0$. In this case the equation (3) reduces to

$$Bx + Cx' + D = 0 \qquad - \quad - \quad - \quad - \quad (23)$$

We may pass over the cases where either $B = 0$ or $C = 0$, which are of no practical interest. They correspond to systems having an infinitely small focal length.

A system in which $A = 0$, $B \neq 0$, $C \neq 0$ we shall call a *Telescopic System.*

The special case where $B = -C$ is worthy of separate consideration. The equation (23) in this case takes the form

$$x' = x - d \qquad - \quad - \quad - \quad - \quad (24),$$

where d is a constant positive or negative according to the nature of the system. The meaning of (24) is that the conjugate focus P′ corresponding to any given point P always lies at a fixed distance d from P in the direction of the passage of the light or in the contrary direction according as d is positive or negative. In particular, we see that to a point at infinity corresponds a point at infinity. Hence *any incident pencil of parallel rays emerges as a parallel pencil. It is obvious in fact, from the more general equation* (23), *that this is a property of any Telescopic System.*

Let Pp (Fig. 9) be any linear object in the plane of the paper perpendicular to the axis of the system and meeting it in P, and let P′p' be the image of Pp. To the incident ray pQ parallel to the axis will correspond the emergent ray p'Q′ also parallel to the axis. Let PQ, P′Q′ be any incident and corresponding emergent rays through P and P′, then Q and Q′ are conjugate foci. Hence, if QM and Q′M′ be perpendicular to the axis, it follows from (24) that

MM′ = d = PP′. Hence PM′ = PM. Now, by Helmholtz's Law, we have

$$\mu\beta\tan\alpha = \mu'\beta'\tan\alpha'$$

Hence $$\mu\beta^2/\mathrm{PM} = \mu'\beta'^2/\mathrm{P'M'};$$

and therefore, since P′M′ = PM,

$$\beta'/\beta = \pm\sqrt{(\mu/\mu')} \qquad - \quad - \quad - \quad (25).$$

In systems of the present kind, therefore, the image is always shifted through a constant distance depending on the nature of the system; the magnification is constant and depends merely on the initial and final media; and the image may be erect or inverted according to the nature of the system.

If the initial and final media be the same, the image is equal to the object. A plane-parallel plate of glass is an example, and lenses can be constructed having the same property, as may easily be seen by working out the characteristic equation (3) for a pair of coaxial spherical surfaces and then applying the conditions A = 0, B + C = 0.

Next suppose B + C ≠ 0. Then, by shifting, the origin through a distance $h = -\mathrm{D}/(\mathrm{B}+\mathrm{C})$, the equation (23) may be reduced to the form

$$x' = kx \qquad - \quad - \quad - \quad - \quad (26),$$

where $k = -\mathrm{B}/\mathrm{C}$. The definite point O which is now the origin we may call *the centre of the telescopic system.*

We see from (26) that the law of conjugate foci reduces now to

[redacted]

Hence $$\beta'/\beta = \pm\sqrt{(\mu k/\mu')} \quad - \quad - \quad - \quad (27);$$
$$\tan\alpha'/\tan\alpha = \pm\sqrt{(\mu/k\mu')} \quad - \quad - \quad - \quad (28).$$

From (27) and (28) we see that the *magnification of the image is constant; and so also is the ratio of the tangents of the inclinations to the axis of any incident and corresponding emergent rays. The image may be erect or inverted according to circumstances.*

We may call the telescopic system Erecting or Inverting according as the image of an infinitely distant point is erect or inverted.

Taking the simple case where the astronomical telescope consists of a field glass of focal length f and an eye piece of focal length g placed at a distance apart equal to $f+g$, if we take the common principal focus for origin, the equation (23) is, if we neglect the thickness of the lenses, and suppose the field-glass turned towards the incident rays,

$$-g^2x+f^2x'-2fg/(f+g)=0 \quad - \quad - \quad - \quad (29).$$

Hence we have

$$h = 2fg/(f-g) \quad - \quad - \quad - \quad - \quad (30);$$
$$h-g=g(f+g)/(f-g) \quad - \quad - \quad - \quad (31).$$

The centre therefore lies at a distance from the eye lens somewhat exceeding its focal length.

From (29) we see that $k=g^2/f^2$. Hence, since, under ordinary circumstances, $\mu=\mu'$, we have

$$\beta'/\beta = \pm g/f \quad - \quad - \quad - \quad (32);$$
$$\tan\alpha'/\tan\alpha = \pm f/g \quad - \quad - \quad - \quad (33):$$

the latter ratio, for reasons into which it is not necessary to enter here, is commonly called the *magnifying power of the telescope.*

Modification of a Symmetrical Photographic Doublet by the introduction of a Thin Lens between its Elements.

For the purposes of landscape photography it is essential to have a series of lenses of widely different focal length. The best possible arrangement would of course be to have a specially constructed lens for each of the focal lengths required; but such a battery of lenses is expensive, and, if doublets are used, it is heavy to carry. Moreover, while the requirements of pictorial perspective absolutely demand variability in the focal length of the photographer's lens, for

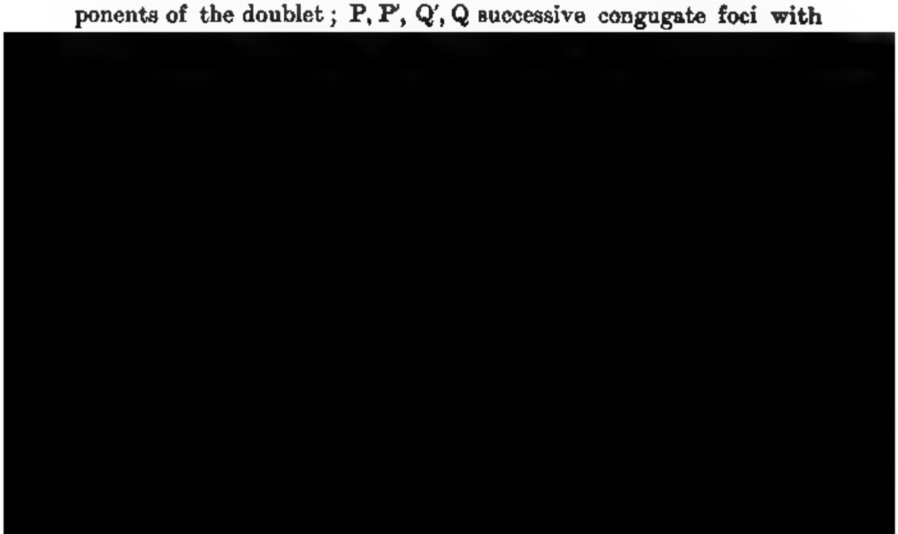
many purposes the utmost refinement in definition and flatness of field is not necessary, or, it may be not desirable.

For general photographic purposes the handiest lens is a symmetrical doublet of the rapid rectilinear or euryscope type; and it has long been known that the focal length of such a combination can be varied within wide limits without destroying its efficiency as a photographic instrument by inserting between its elements a thin lens of positive or negative focal length. As an example of the foregoing general theory, we propose to calculate the effect of such a lens in shifting the principal points and in altering the focal length of the doublet.

We shall suppose the thickness of the inserted lens to be negligible; *i.e.*, we shall take its principal points to be coincident with the middle point of the substance of the lens; and in the first instance we shall suppose the inserted lens to be actinically achromatised, so that it has no "focal difference," in other words that its principal foci for the rays of maximum visual and maximum chemical intensity coincide.

Let f (Fig. 11) be the focal length of each of the components of the doublet, F and ϕ the focal lengths of the doublet itself and of the adjuster respectively, we take ϕ to be positive as usual when the adjuster is of positive focal length.

Let L (Fig. 11) be the position of the adjuster; C the central point of the doublet; H, H′, K, K′ the principal points of the components of the doublet; P, P′, Q′, Q successive congugate foci with



If we put $(h'-h)f+hh'=k^2, f+h=l, f-h'=m,$
so that $k^2=(2f-l-m)f+(f-m)(l-f)=f^2-lm,$ } (38)

then we may replace h, h', f by $k, l, m,$ or by l, m, f; where it will be observed that l and m are the distances from the central point of the doublet of the outer and inner principal focal points of either of its elements.

If now we substitute the expressions just found for $x'+d$ and $y'-d$ in (36), we get

$$\phi(l-x)\{k^2-dl+(m+d)y\}+\phi(l-y)\{k^2+dl+(m-d)x\}$$
$$=\{k^2+dl+(m-d)x\}\{k^2-dl+(m+d)y\}.$$

Hence the characteristic equation for the triplet is

$$Axy+By+Cx+D=0 \qquad (39),$$

where

$$\left.\begin{aligned} A&=2m\phi+m^2-d^2,\\ B&=(k^2+dl)(m+d)+(k^2-lm)\phi,\\ C&=(k^2-dl)(m-d)+(k^2-lm)\phi,\\ D&=k^4-d^2l^2-2k^2l\phi. \end{aligned}\right\} \qquad (40).$$

For the coordinates of the principal focal points of the triplet we have

$$\left.\begin{aligned} g&=-\frac{B}{A}=\frac{(lm-k^2)\phi-(k^2+dl)(m+d)}{2m\phi+m^2-d^2};\\ g'&=-\frac{C}{A}=\frac{(lm-k^2)\phi-(k^2-dl)(m-d)}{2m\phi+m^2-d^2} \end{aligned}\right\} \qquad (41).$$

Also, if F_3 be the principal focal length of the triplet,

$$F_3^2=\frac{BC-AD}{A^2}=\left(\frac{(k^2+lm)\phi}{2m\phi+m^2-d^2}\right)^2;$$

whence

$$F_3=\frac{(k^2+lm)\phi}{2m\phi+m^2-d^2}=\frac{f^2\phi}{2m\phi+m^2-d^2} \qquad (42).$$

If $\phi=\infty$, $F_3=F$; so that $F=f^2/2m$, and we may put the last equation into the form

$$F_3=\frac{F\phi}{\phi+\frac{1}{2}m-d^2/2m} \qquad (43).$$

If χ, χ' be the coordinates of the principal points of the triplet, we have

$$\left.\begin{aligned} \chi &= g - F_3 = -\frac{2k^2\phi + (k^2 + dl)(m+d)}{2m\phi + m^2 - d^2}\,; \\ \chi' &= g' - F_3 = -\frac{2k^2\phi + (k^2 - dl)(m-d)}{2m\phi + m^2 - d^2} \end{aligned}\right\} \quad (44).$$

Case where the Adjuster is Central.

Then $d = 0$; and we have $F_3 = F\phi/(\phi + \frac{1}{2}m)$ - - - (45);

whence $\phi = \frac{1}{2}mF_3/(F - F_3)$ - - - (46):

these formulæ give the focal length of the triplet corresponding to a given adjuster and the focal length of the adjuster required to produce a triplet of given focal length.

The coordinates of the principal points now become

$$\chi = \chi' = -k^2/m = l - f^2/m = f + h - 2F \quad - \quad (47):$$

Since these are the values of χ and χ' given by the general formulæ (44) when $\phi = \infty$, we see that a central adjuster leaves the principal points of the doublet unaltered.

Non-Central Adjuster.

[redacted]

while each is shifted towards the inserted lens through a distance λ which is given approximately by

$$\lambda = \mathrm{F}d/(\phi + \tfrac{1}{2}m) \qquad - \quad - \quad - \quad (48);$$

or

$$\lambda = d(\mathrm{F} - \mathrm{F}_3)/\tfrac{1}{2}m \qquad - \quad - \quad - \quad (49)$$

The result just stated may be put into another form. Reverting to the expression for F_3 when $d = 0$, we have

$$\frac{1}{\mathrm{F}_3} = \frac{1}{\mathrm{F}} + \frac{m}{2\mathrm{F}\phi} = \frac{1}{\mathrm{F}} + \frac{f - h'}{2\mathrm{F}\phi}.$$

Now

$$\frac{1}{\mathrm{F}} = \frac{2}{f} - \frac{2h'}{f^2}.$$

Hence

$$\frac{1}{f} = \frac{1}{2\mathrm{F}} + \frac{h'}{f^2}.$$

A first approximation to the value of f is $f = 2\mathrm{F}$
For a second approximation

$$\frac{1}{f} = \frac{1}{2\mathrm{F}} + \frac{h'}{4\mathrm{F}^2};$$

and for a third approximation

$$\frac{1}{f} = \frac{1}{2\mathrm{F}}\left\{1 + \frac{h'}{2\mathrm{F}} + \frac{h'^2}{2\mathrm{F}^2}\right\}.$$

Hence, to a third approximation, we get

$$f - h' = 2\mathrm{F}\left\{1 - \frac{h'}{\mathrm{F}} - \frac{h'^2}{4\mathrm{F}^2}\right\}.$$

To the same degree of approximation

$$\frac{1}{\mathrm{F}_3} = \frac{1}{\mathrm{F}} + \frac{1}{\phi} - \frac{h'}{\phi\mathrm{F}} - \frac{h'^2}{4\mathrm{F}^2\phi}.$$

If, therefore, $h^2/4\mathrm{F}^2\phi$ be negligible, we get

$$\frac{1}{\mathrm{F}_3} = \frac{1}{\mathrm{F}} + \frac{1}{\phi} - \frac{h'}{\mathrm{F}\phi}. \qquad (50).$$

In other words, the triplet behaves, *qua* focal length, as if the doublet were replaced by a thin lens of its own focal length placed at its centre and the adjuster were placed at the inner principal point of one of its components. This agrees with what we have

already said regarding the indifference of the position of the adjuster so far as the focal length of the triplet is concerned.

When a negative adjuster is used to lengthen the focus of the doublet, there is a practical advantage in placing it as near the back component as possible, because the effect of this is to shift the principal points forward (λ being negative) so that less camera extension is required. Thus, for example, my camera scarcely allows me to use the back lens ($f = 18{\cdot}06''$) of my Voigtländer's Euryscope as a single landscape lens, whereas I can readily use the Euryscope adjusted to a focus of 20″ by inserting an adjuster for which $\phi = -16$ near the back lens.

In practice the simplest method for obtaining data for the construction of adjusters of a given symmetrical doublet is to measure the focal length of the doublet itself; this gives F; then to measure the distance behind the central point of the doublet (usually the place where the diaphragm is put) of the inner principal focal point of its front element; this gives m. The formulæ (45) (46) will then give F, or ϕ as may be required. If there is any reason to doubt the accuracy of the approximation when the adjuster is non-central, the more accurate formula (43) may be used.

The following table gives the results of several (visual) experiments made to test the foregoing results. The doublet was the Euryscope above mentioned; and the formula used for calculating F_3 was $F_3 = 9{\cdot}85\phi/(\phi + 8{\cdot}2)$.

[redacted]

Chromatic Aberration Caused by a Non-Achromatic Adjuster.

Hitherto the adjusting lens has been supposed to be actinically achromatic. As a matter of fact, the lenses used in my experiments were simple lenses of crown glass. It is easy to calculate, by means of the formulæ given above, the chromatic aberration or actinic focal difference produced by the non-achromatic adjuster. The elements of the Euryscope itself are approximately corrected for actinic chromatism: We have therefore only to deal with the dispersion of the adjuster itself. Assuming the lens to be of hard crown glass, and taking the rays of maximum visual and maximum chemical intensity respectively to be D and G, we may suppose $\mu_D = 1{\cdot}5167$, $\mu_G = 1{\cdot}5280$: hence $\omega = \partial\mu/(\mu_D - 1) = {\cdot}022$, say.

From the approximate formulæ for the triplet we have

$$1/F_3 = 1/F + \tfrac{1}{2}m/F\phi,\ \lambda = d(F - F_3)/\tfrac{1}{2}m.$$

Hence*
$$\partial\left(\frac{1}{F_3}\right) = \frac{\frac{1}{2}m}{F}\,\partial\left(\frac{1}{\phi}\right) = \frac{\frac{1}{2}m\omega}{F\phi}:$$

therefore
$$\partial F_3 = -\tfrac{1}{2}mF_3\omega/\phi; \tag{51}$$
$$\partial\lambda = -d\partial F_3/\tfrac{1}{2}m = dF_3\omega/\phi. \tag{52}$$

Let us suppose the adjuster placed behind the centre of the doublet as before; let u and v be the distance of object and image (corresponding to the ray D of the spectrum) from the first and second principal points of the triplet respectively; and let us calculate the longitudinal aberration of the ray G. This is caused partly by the shifting of the principal points and partly by the alteration of the focal length. The first principal point is shifted to the right through $\partial\lambda$; and u as measured from the new first principal point is increased by the same amount: v as measured from the new second principal point is increased by an amount ∂v which is given by

$$\partial\lambda/u^2 + \partial v/v^2 = \partial F_3/F_3^2,$$

that is, by (51) and (52)

$$\begin{aligned}\partial v &= v^2\partial F_3/F_3^2 - v^2\partial\lambda/u^2,\\ &= -\tfrac{1}{2}mv^2\omega/F_3\phi - dv^2F_3\omega/u^2\phi.\end{aligned} \tag{53}$$

* See Heath, § 182.

To this must be added the shift, $\partial\lambda$, of the second principal point to the right. If, therefore, α denote the whole longitudinal aberration of the ray G, we have

$$\alpha = \frac{\omega F_2 d}{\phi} - \frac{\omega v^2 \frac{1}{2} m}{F_2 \phi} - \frac{\omega v^2 F_2 d}{u^2 \phi} \quad - \quad - \quad - \quad (54).$$

If we consider only the correction for very distant objects, we may put $u = \infty$, $v = F_2$: we then get

$$\alpha = \omega(d - \tfrac{1}{2}m)F_2/\phi \quad - \quad - \quad - \quad (55).$$

Taking $d = \cdot 6''$, $\frac{1}{2}m = 8\cdot 2''$, and $\omega = \cdot 022$, we find

$$\alpha = -\cdot 167 F_2/\phi \quad - \quad - \quad - \quad (56),$$

by means of which we can calculate the actinic focal difference for any given adjuster. Thus, for example, when $\phi = -15\cdot 89''$ and $F_2 = 20\cdot 35''$, $\alpha = +\cdot 22''$; that is to say, the camera must be racked out a little over $\cdot 2''$ after the view has been focussed on the ground glass.

Experimental Determination of the Constants of a Symmetrical System.

Since this paper is intended mainly for the use of laboratory students, a word or two on the experimental determination of the characteristic points of a symmetrical optical system may not be [redacted]

If now we determine the position P′ of any axial object P, we can find FP and F′P′ and hence f, by the equation $FP \cdot F'P' = f^2$ and it only remains to note whether the system is erecting or inverting in order to be able to lay down the principal and anti-principal points H, H′ and K, K′. In the case of a convex lens, for example, this is readily done by a method given by Gauss, which consists in focussing a microscope on an object in the axis of the lens in contact with its surface, first through the lens and then after the lens has been removed: the amount by which the draw-tube has been displaced between the two focussings is the distance between P and P′, from which FP and FP′ can be found.

We may also determine the principal or the antiprincipal points directly. For this purpose two identical photographic negatives O and O′ of the same object, say a divided scale, may be used. O is placed perpendicular to the axis of the system and its image received on O′ similarly placed as a focussing screen, so that the two scales overlap. When the image of O is exactly of the same size as O′, O and O′ are in the positions H, H′ or else K, K′ respectively. The coincidence may be very accurately determined by observing O′ and the image of O by means of a microscope or other magnifier of moderate power carefully focussed on O′ beforehand.

These measurements are susceptible of considerable accuracy if monochromatic light of sufficent intensity be used, and the experimenter is provided with an optic bench fitted with telescope and low power microscope with micrometer eyepiece and micrometer displacement screw, together with arrangements for fixing and centering the systems to be measured.

The observations quoted above, which are probably accurate to about 1%, were made in my own library, with white light, the only apparatus available being a small pocket telescope, the *debris* of a toy microscope, a steel measuring tape and a couple of retort stands.

Address on Celestial Photography

(With Lantern Illustrations).

By Mr Alex. Morgan, M.A., B.Sc.

Second Meeting, December 13th, 1895.

Wm. Peddie, Esq., D.Sc., F.R.S.E., President, in the Chair.

Note on the Circles of Curvature of a Plane Curve.

By Professor Tait.

When the curvature of a plane curve continuously increases or diminishes (as is the case with a logarithmic spiral, for instance) no two of its circles of curvature can intersect one another.

This curious remark occurred to me some time ago in connection

[redacted]

A Reduction Formula for Indefinite Integrals.

BY GEORGE A. GIBSON.

On p. 403 of Greenhill's *Calculus* (2nd Ed.) the following sentence occurs :—" By differentiation of the integral

$$\int \frac{Hx+K}{Ax^2+2Bx+C} \frac{dx}{\sqrt{(ax^2+2bx+c)}}$$

with respect to A, B, or C we can deduce the results of

$$\int \frac{f(x)}{(Ax^2+2Bx+C)^n} \frac{dx}{\sqrt{(ax^2+2bx+c)}} \text{"}$$

For the evaluation of the typical form in which $f(x)$ is a linear function, especially when A, B, etc., are given numbers, the method of differentiation does not seem very suitable; be that as it may, it may perhaps be of some interest to investigate a formula of reduction analogous to those in use for the integrals in which $Ax^2+2Bx+C$ is replaced by a linear function and $f(x)$ is a constant. I had occasion lately, in class work, to use such a reduction formula; but so far as I can find none of our text-books contains the reduction, except Robert's *Integral Calculus*, § 50. I propose to apply the method so beautifully used by Hermite in his *Cours* (4ième Éd., Léçon IV.) to effect the reduction in the case mentioned. It should be stated, however, that every method seems likely to be very laborious in practice, even when the constants are small numbers. At the same time the reduction is of great theoretical interest if the use of complex constants is to be avoided in a course on the *Integral Calculus*.

Let $$R=ax^2+2bx+c, \quad S=Ax^2+2Bx+C$$

$$S'=\frac{dS}{dx}=2(Ax+B), \qquad T=\tfrac{1}{2}RS'$$

S is supposed to have only complex factors.

If T be prime to S and to $Hx+K$, then by the theory of partial fractions

$$\frac{Hx+K}{ST}=\frac{Px+Q}{S}+\frac{Lx^2-Mx-N}{T}$$

where obviously $L=-aP$, and the constants have to be determined in the usual way. Hence

$$Hx+K=(Px+Q)T-(aPx^2+Mx+N)S.$$

Substituting this value of $Hx+K$ in the integral we get

$$\int\frac{Hx+K}{S^n\sqrt{R}}dx=\tfrac{1}{2}\int\frac{(Px+Q)\sqrt{R}}{S^n}S'dx-\int\frac{aPx^2+Mx+N}{S^{n-1}\sqrt{R}}dx.$$

Integrating by parts the first integral on the right we have

$$\int\frac{Hx+K}{S^n\sqrt{R}}dx=\frac{-1}{2n-2}\quad\frac{(Px+Q)\sqrt{R}}{S^{n-1}}-\frac{1}{2n-2}\int\frac{F(x)dx}{S^{n-1}\sqrt{R}}$$

where

$$F(x)=(2n-4)aPx^2+\{(2n-2)M-3bP-aQ\}x+(2n-2)N-bQ.$$

If $n=2$, $F(x)$ is linear and the formula of reduction has been obtained; but if $n\neq 2$, we may write

[redacted]

If it should happen that $Hx+K$ is not prime to T, then (i.) $Hx+K=k(Ax+B)$, (ii.) $Hx+K$ is a factor of R, say $R=(Hx+K)(a'x+b')$. In case (i.) we have

$$\frac{k}{SR}=\frac{px+q}{S}+\frac{lx+m}{R}$$

or
$$k=(px+q)R+(lx+m)S$$

and
$$Hx+K=\tfrac{1}{2}(px+q)RS'+(lx+m)(Ax+B)S$$

and we proceed as before. In case (ii.)

$$\frac{1}{S(Ax+B)(a'x+b')}=\frac{(px+q)}{S}+\frac{lx+m}{(Ax+B)(a'x+b')}$$

and
$$Hx+K=\tfrac{1}{2}(px+q)RS'+(lx+m)(Hx+K)S$$

If R were not prime to S, this would mean $S=kR$ and the integral then belongs to a well-known form.

In applying the method to a numerical example the first step is to find P, Q; we may take a very simple case by way of illustration.

Let
$$u=\int\frac{2x+1}{(x^2+2x+4)^n}\frac{dx}{\sqrt{(x^2-2x+3)}}$$

By decomposing
$$\frac{2x+1}{(x^2+2x+4)(x+1)(x^2-2x+3)}$$

we get
$$57(2x+1)=(9x+11)(x+1)R-(9x^2-16x-6)S$$

so that

$$57u=\frac{-(9x+11)\sqrt{R}}{(2n-2)S^{n-1}}-\int\frac{(2n-4)9x^2-(2n-3)16x-12n-4}{(2n-2)S^{n-1}\sqrt{R}}dx$$

$$=\frac{-(9x+11)\sqrt{R}}{(2n-2)S^{n-1}}+\int\frac{(34n-60)x+42n-70}{(n-1)S^{n-1}\sqrt{R}}dx$$

$$-\frac{9(n-2)}{n-1}\int\frac{dx}{S^{n-2}\sqrt{R}}$$

If $n=2$, this gives

$$57u=\frac{-(9x+11)\sqrt{R}}{2S}+\int\frac{8x+14}{S\sqrt{R}}dx$$

To integrate $(8x+14)/S\sqrt{R}$ we may take Greenhill's method; or we may use the bilinear substitution $x=(py+q)/(y+1)$ and choose p, q (which can always be done) so as to make the integral take the form

$$\int\frac{Hx+K}{Ax^2+C}\quad\frac{dx}{\sqrt{(ax^2+c)}}$$

i.e.
$$H\int\frac{xdx}{(Ax^2+C)\sqrt{(ax^2+c)}}+K\frac{dx}{(Ax^2+C)\sqrt{(ax^2+c)}}$$

The first of these is integrated by putting $ax^2+c=z^2$ and the second by putting $a+cx^{-2}=z^2$.

Some Properties of Parabolic Curves.

By George A. Gibson, M.A.

If the tangent at a point P on the parabolic curve $cy = x^n$ meet the axis of x at M, it is a well-known property that the area between the radius vector OP and the arc OP is n times that between the arc OP and the two tangents OM, MP, O being the origin and $n > 1$. The converse is also true; for taking any point O on a curve as origin and the tangent at O as axis of x, let us seek for the locus of P if the area between OP and the arc OP be n times the area between the arc OP and the tangents OM, MP.

The area between the chord OP and the arc OP is

$$\tfrac{1}{2}xy - \int_0^x ydx$$

and the area between arc and tangents is

$$\int_0^x ydx - \frac{y^2}{2p}$$

where $p = dy/dx$. Hence

$$\tfrac{1}{2}xy - \int_0^x ydx = n\int_0^x ydx - \frac{ny^2}{2p}$$

Differentiating with respect to x, the differential equation of the curve will be

$$\frac{ny^2}{p^2}\,\frac{dp}{dx} = xp - y$$

This may be written

$$n\frac{d}{dx}\left(\frac{1}{p}\right) = \frac{d}{dx}\left(\frac{x}{y}\right)$$

$$\therefore \frac{n}{p} = \frac{x}{y} + C$$

$$i.e. \ \frac{dx}{dy} - \frac{1}{ny}x = \frac{C}{n}$$

the integral of which is

$$x = \mathrm{D}y^{\frac{1}{n}} + \frac{\mathrm{C}}{n-1}y$$

or

$$y = (\mathrm{A}x + \mathrm{B}y)^n$$

If $\mathrm{B} = 0$, we have the form $cy = x^n$; and in general if $\mathrm{A}x + \mathrm{B}y = 0$ be taken as axis of y in a system of oblique coordinates, the equation takes the same form $cy = x^n$.

If n were a positive proper fraction, the axes would simply be interchanged.

Consider more particularly the parabola $x^2 = 4ay$. In this case the area between OP and the curve is $b^3/24a$ if b is the abscissa of P, while the area between the arc and the tangents is $b^3/48a$. It will be noticed that $b^3/48a$ is the area between the chord OP′ and the arc OP′ of the parabola $x^2 + ay = 0$ where $b/2$ is the abscissa of P′. But $b/2$ is the abscissa of M while the ordinate of $x^2 + ay = 0$ for the abscissa $b/2$ is $-b^2/4a$, that is, the intercept made by the tangent at P on the axis of y. In fact $x^2 + ay = 0$ is the locus of a point which has for coordinates the intercepts made by the tangent at P on the axis of x and y. (Compare Forsyth's *Diff. Equations*, p. 41 ex. 9.) How far does this property hold for the general parabola? In other words what is the solution of the following

[redacted]

The area cut off by the chord OP′ from the locus of P is

$$\int_0^{\xi} \eta d\xi + \frac{y^2}{2p} + \tfrac{1}{2}px^2 - xy$$

both areas being positive. Hence

$$n\int_0^{x} ydx - \frac{ny^2}{2p} = \int_0^{\xi} \eta d\xi + \frac{y^2}{2p} + \tfrac{1}{2}px^2 - xy$$

Differentiating with respect to x and noting that

$$\frac{d\xi}{dx} = \frac{y}{p^3}\,\frac{dp}{dx}$$

we get
$$\left(\frac{n-1}{2}\,\frac{y^2}{p^2} + \frac{xy}{p} - \tfrac{1}{2}x^2\right)\frac{dp}{dx} = 0$$

$dp/dx = 0$ gives no solution. Hence the equation of the locus of P is given by

$$p^2x^2 - 2xyp - (n-1)y^2 = 0$$

or
$$xp = y(1 \pm \sqrt{n})$$

the integral of which is $cy = x^{1\pm\sqrt{n}}$, giving only one solution, $cy = x^2$ when $n = 1$.

If n be not a square each curve is transcendental, but if $n = m^2$, we have $cy = x^{m+1}$ or $cy = x^{1-m}$. The solution $cy = x^{1-m}$ evidently does not satisfy the conditions of the problem, the axis of x not being the tangent at 0, but obviously the other solution $cy = x^{m+1}$ does.

To find the locus of P′ we have

$$\xi = \frac{m}{m+1}x, \quad \eta = -\frac{m}{c}x^{m+1}$$

and therefore
$$\xi^{m+1} + \frac{cm^m}{(m+1)^{m+1}}\eta = 0$$

These are parabolic curves which for $m = 1$ reduce to the ordinary parabola.

With regard to the solution $cy = x^{1-m}$, it may be noted that when m is greater than two the axes are asymptotes and a similar proposition holds for the two loci. Using the form $x^{m-1}y = k$ as the equation to the locus of P we find for the locus of P′ the equation

$$\xi^{m-1}\eta = \frac{km^m}{(m-1)^{m-1}}$$

The area bounded by the tangent PM, the part of the axis of x from M to $+\infty$ and the arc from P to the same end of the axis of x is

$$\frac{mxy}{2(m-1)(m-2)}$$

On the other hand the area bounded by the line OP′, the positive part of the axis of x and the arc of the locus of P′ from P′ to the positive end of the axis of x is

$$\frac{m\xi\eta}{2(m-2)} = \frac{m^2xy}{2(m-1)(m-2)}$$

and is therefore m^2, *i.e.*, n times the former area.

When m is less than 1 the tangent at the origin to the curve $cy = x^{1-m}$ is the axis of y and a similar proposition to that given for the curve $cy = x^{m+1}$ holds if M and N be taken on the axes of y and x respectively, while if m be greater than 1 but less than 2 the same

[redacted]

Development of *sn x*, *cn x*, *dn x*, by means of their addition theorems.

By J. Jack, M.A.

Taking the three addition theorems and clearing away the fractions,

$$sn\ \overline{x+y} = \begin{cases} \sin x\, cn\, y\, dn\, y + sn\, y\, cn\, x\, dn\, x \\ + k^2 \sin^2 x \sin^2 y \sin \overline{x+y} \end{cases} \tag{1}$$

$$cn\ \overline{x+y} = \begin{cases} cn\, x\, cn\, y - sn\, x \sin y\, dn\, x\, dn\, y \\ + k^2 \sin^2 x \sin^2 y \cos \overline{x+y} \end{cases} \tag{2}$$

$$dn\ \overline{x+y} = \begin{cases} dn\, x\, dn\, y - k^2\, sn\, x \sin y\, cn\, x\, cn\, y \\ + k^2 \sin^2 x \sin^2 y\, dn\, \overline{x+y} \end{cases} \tag{3}$$

Let

$$\begin{aligned} sn\ x &= a_1 x + a_3 x^3 + a_5 x^5 + \dots \\ cn\ x &= a_0 + a_2 x^2 + a_4 x^4 + a_6 x^6 + \dots \\ dn\ x &= b_0 + b_2 x^2 + b_4 x^4 + b_6 x^6 + \dots \end{aligned}$$

Substitute in (1) (2) (3) the expansions for $\sin x$, $\sin y$, $\sin \overline{x+y}$ $\cos x$, $\cos y$, $\cos \overline{x+y}$, $dn\, x$, $dn\, y$, $dn\, \overline{x+y}$, and then pick out the coefficients of y in (1), (2), (3). Then equate the like powers of x in the resulting series.

From (1)

$$\begin{aligned} a_1 &= a_0 b_0 a_1 \\ 3a_3 &= a_1(a_0 b_2 + a_2 b_0) \\ 5a_5 &= a(a_0 b_4 + a_2 b_2 + a_4 b_0) \end{aligned}$$

and so on.

From (2) $\qquad a_0 = a_0^2$

$$-2a_2 = a_1 b_0 (a_1 b_0)$$

$$-4a_4 = a_1 b_0 (a_1 b_2 + a_3 b^0)$$

$$-6a_6 = a_1 b_0 (a_1 b_4 + a_3 b_2 + a_5 b_0)$$

and so on.

From (3) $\qquad b_0 = b_0^2$

$$-2b_2 = k^2 a_1 a_0 (a_1 a_0)$$

$$-4b_4 = k^2 a_1 a_0 (a_1 a_2 + a_3 a_0)$$

$$-6b_6 = k^2 a_1 a_0 (a_1 a_4 + a_3 a_2 + a_5 a_0)$$

and so on.

Hence $\qquad a_0 = 1, \qquad b_0 = 1, \qquad a_1$ *undetermined,*

$$a_2 = -\frac{a_1^2}{1.2}, \quad b_2 = -\frac{k^2 a_1^2}{1.2},$$

$$a_3 = \tfrac{1}{3} a_1 (a_2 + 6_2) = \frac{-a_1^3(1+k^2)}{1.2.3}$$

Similarly $\qquad a_4 = \dfrac{a_1^4(1+4k^2)}{1.2.3.4}, \quad b_4 = \dfrac{a_1^4 k^2 (4+k^2)}{1.2.3.4},$

[redacted]

Third Meeting, January 10th, 1896.

Dr PEDDIE in the Chair.

Symmedians of a Triangle and their concomitant Circles

BY J. S. MACKAY, M.A., LL.D.

NOTATION

A B C	= vertices of the fundamental triangle
A′ B′ C′	= mid points of BC CA AB
D E F	= points of contact of sides with incircle
	= other triads of points defined as they occur
D_1 E_1 F_1	= points of contact of sides with first excircle
	And so on
G	= centroid of ABC
I	= incentre of ABC
I_1 I_2 I_3	= 1st 2nd 3rd excentres of ABC
J, J_1 J_2 J_3	= quartet of points defined in the text
K	= insymmedian point of ABC
K_1 K_2 K_3	= 1st 2nd 3rd exsymmedian points of ABC
L M N	= projections of K on the sides of ABC
L_1 M_1 N_1	= „ „ K_1 „ „ „
	And so on
O	= circumcentre of ABC
R S T	= feet of the insymmedians
R′ S′ T′	= „ „ exsymmedians
X Y Z	= „ „ perpendiculars from A B C

INTRODUCTORY

Definition. The isogonals * of the medians of a triangle are called the *symmedians* †

If the internal medians be taken, their isogonals are called the *internal symmedians* ‡ or the *insymmedians*, if the external medians be taken, their isogonals are called the *external symmedians*, or the *exsymmedians*

The word symmedians, used without qualification or prefix, may, as in the title of this paper, be regarded as including both insymmedians and exsymmedians (*cyclists* include both *bicyclists* and *tricyclists*); frequently however when used by itself it denotes insymmedians, just as the word medians denotes internal medians

It is hardly necessary to say that as medians and symmedians are particular cases of isogonal lines, the theorems proved regarding the latter are applicable to the former. Medians and symmedians however have some special features of interest, which are easier to examine and recognise than the corresponding ones of the more general isogonals

Definition. Two points D D′ are *isotomic* § with respect to BC when they are equidistant from the mid point of BC

It is a well-known theorem (which may be proved by the theory of transversals) that

If three concurrent straight lines AD BE CF be drawn from the vertices of ABC to meet the opposite sides in D E F, and if D′ E′ F′ be isotomic to D E F with respect to BC CA AB, then

[redacted]

Instead of saying that D D′ are isotomic points with respect to BC, it is sometimes said that AD AD′ are isotomic lines with respect to angle A

§ 1

Construction for an insymmedian

FIGURE 12

Let ABC be the triangle

Draw the internal median AA′ to the mid point of BC; and make $\angle$ B A R = $\angle$ CAA′

AR is the insymmedian from A

The angle C A A′ is described clockwise, and the angle B A R counter-clockwise;
consequently AA′ AR are symmetrically situated with respect to the bisector of the interior angle B A C

Hence since AA′ is situated inside triangle ABC,
AR is inside ABC

The following construction* leads to a simple proof of a useful property of the insymmedians

FIGURE 13

From AC cut off AB_1 equal to AB
and " AB " " AC_1 " " AC
If B_1C_1 be drawn, it will intersect BC at L the foot of the bisector of the interior angle B A C

Hence if A A′, which is obtained by joining A to the point of internal bisection of BC, be the internal median from A, the corresponding insymmedian A R is obtained by joining A to the point of internal bisection of B_1C_1

* Mr Maurice D'Ocagne in *Journal de Mathématiques Élémentaires et Spéciales*, IV. 539 (1880). This construction, which recalls Euclid's *pons asinorum*, is substantially equivalent to a more complicated one given by Const. Harkema of St Petersburg in Schlömilch's *Zeitschrift*, XVI. 168 (1871)

(1) BC and B_1C_1 are antiparallel with respect to angle A

(2) Since the internal median AA′ bisects internally all parallels to BC, therefore the insymmedian AR bisects internally all antiparallels to BC

(3) The insymmedians of a triangle bisect the sides of its orthic triangle *

(4) *The projections of B and C on the bisector of the interior angle BAC are P and Q. If through P a parallel be drawn to AB, and through Q a parallel be drawn to AC, these parallels will intersect † on the insymmedian from A*

[The reader is requested to make the figure]

Let A″ be the point of intersection of the parallels, and A′ the mid point of BC

It is well known ‡ that A′P is parallel to AC, that A′Q is parallel to AB, and that

$$A'P = \tfrac{1}{2}(AC - AB) = A'Q$$

Hence the figure A′PA″Q is a rhombus,

and A″ is the image of A′ in the bisector of angle A

Now since A′ lies on the median from A

[redacted]

The symmedian point has three points harmonically associated with it; when it is necessary to distinguish it from them, the name insymmedian point will be used

The insymmedian point and the centroid of a triangle are *isogonally conjugate points*

(6) If XYZ be the orthic triangle of ABC the insymmedian points of the triangles AYZ XBZ XYC are situated on the medians* of ABC

(7) *The insymmedian from the vertex of the right angle in a right-angled triangle coincides with the perpendicular from that vertex to the hypotenuse*†, *and the three insymmedians intersect at the mid point of this perpendicular* ‡

The first part of this statement is easy to establish. The second part follows from the fact that the orthic triangle of the right-angled triangle reduces to the perpendicular

§ 1′

Construction for an exsymmedian

FIGURE 14

Let ABC be the triangle

Draw the external median AA_∞ parallel to BC;
and make $\angle BAR' = \angle CAA_\infty$

AR′ is the exsymmedian from A

The angle CAA_∞ is described counterclockwise, and the angle BAR′ clockwise;
consequently AA_∞ AR′ are symmetrically situated with respect to the bisector of the exterior angle BAC

Hence since AA_∞ is situated outside triangle ABC,
AR′ is outside ABC

The following construction leads to a simple proof of a useful property of the exsymmedians

* Dr Franz Wetzig in Schlömilch's *Zeitschrift*, XII. 288 (1867)

† C. Adams's *Eigenschaften des...Dreiecks*, p. 2 (1846)

‡ Mr Clément Thiry's *Le troisième livre de Géométrie*, p. 42 (1887)

FIGURE 15

From CA cut off AB_1 equal to AB
and „ BA „ „ AC_1 „ „ AC
If B_1C_1 be drawn, it will intersect BC at L′ the foot of the bisector of the exterior angle BAC

Hence if AA_∞, which is obtained by joining A to the point of external bisection of BC (that is, by drawing through A a parallel to BC) be the external median from A, the corresponding exsymmedian AR′ is obtained by joining A to the point of external bisection of B_1C_1 (that is, by drawing through A a parallel to B_1C_1)

(1′) BC and B_1C_1 are antiparallel with respect to angle A

(2′) Since the external median AA_∞ bisects externally all parallels to BC, therefore the exsymmedian AR′ bisects externally all antiparallels to BC

(3′) The exsymmedians of a triangle are parallel to the sides of its orthic triangle*

(4′) *The projections of B and C on the bisector of the exterior angle BAC are P′ and Q′. If through P′ a parallel be* [redacted]

Three points are thus obtained, and they are sometimes called the *exsymmedian* points

The three points obtained by the intersections of the external medians of ABC are the vertices of the triangle formed by drawing through A B C parallels to BC CA AB; that is, they are the points anticomplementary* to A B C

Hence the exsymmedian points of a triangle are *isogonally conjugate* to the anticomplementary points of the vertices of the triangle

(6′) *The tangents to the circumcircle of a triangle at the three vertices are the three exsymmedians of the triangle*†

FIGURE 14

For $\angle BAR' = \angle CAA_\infty$

$= \angle ACB$

therefore AR′ touches the circle ABC at A

(7′) When the triangle is right-angled two of the exsymmedians are parallel, or they intersect at infinity on the perpendicular drawn from the vertex of the right angle to the hypotenuse

§ 2

The distances of any point in an insymmedian from the adjacent sides are proportional to those sides‡

FIGURE 16

Let AA′ be the internal median, AR the insymmedian from A

From R draw RV RW perpendicular to AC AB;

and „ A′ „ A′P A′Q „ „ „ „

Then $RW : RV = A'P : A'Q$

$= AB : AC$

* See *Proceedings of the Edinburgh Mathematical Society*, I. 14 (1894)

† C. Adams's *Eigenschaften des... Dreiecks*, p. 5 (1846)

‡ Ivory in Leybourn's *Mathematical Repository*, new series, Vol. I. Part I. p. 26 (1804). Lhuilier in his *Élémens d'Analyse*, p. 296 (1809) proves that

$RW : RV = \sin C : \sin B$

§ 2′

The distances of any point in an exsymmedian from the adjacent sides are proportional to those sides

FIGURE 17

Let AA_o be the external median, AR' the exsymmedian from A

From R' draw $R'V'$ $R'W'$ perpendicular to AC AB;
and from A_1 any point in the external median, draw A_1P' A_1Q' perpendicular to AC AB

Then $$R'W' : R'V' = A_1P' : A_1Q'$$
$$= AB : AC$$

§ 3

*The segments into which an insymmedian from any vertex divides the opposite side are proportional to the squares of the adjacent sides**

FIGURE 16

Let AR be the insymmedian from A

[redacted]

§ 3′

*The segments into which an exsymmedian from any vertex divides the opposite side are proportional to the squares of the adjacent sides**

FIGURE 17

Let AR′ be the exsymmedian from A
Draw R′V′ R′W′ perpendicular to AC AB

Then $$AB : AC = R'W' : R'V'$$

therefore
$$\begin{aligned} AB^2 : AC^2 &= AB \cdot R'W' : AC \cdot R'V' \\ &= ABR' : ACR' \\ &= BR' : CR' \end{aligned}$$

§ 4

The insymmedians of a triangle are concurrent

FIRST DEMONSTRATION

FIGURE 18

Let AR BS CT be the insymmedians

Then
$$\begin{aligned} BR : CR &= AB^2 : AC^2 \\ CS : AS &= BC^2 : BA^2 \\ AT : BT &= CA^2 : CB^2 \end{aligned}$$

therefore
$$\frac{BR}{CR} \cdot \frac{CS}{AS} \cdot \frac{AT}{BT} = -1$$

since of the ratios BR : CR CS : AS AT : BT all are negative; therefore AR BS CT are concurrent

---

* C. Adams's *Eigenschaften des ... Dreiecks*, pp. 3-4 (1846). Pappus in his *Mathematical Collection*, VII. 119 gives the following theorem as a lemma for one of the propositions in Apollonius's *Loci Plani*:

If $AB^2 : AC^2 = BR' : CR'$
then $BR' \cdot CR' = AR'^2$

Second Demonstration

Figure 19

Let BK CK the insymmedians from B C cut each other at K: to prove that K lies on the insymmedian from A

Through K draw

EF′	antiparallel	to BC	with	respect	to A
FD′	,,	,, CA	,,	,,	,, B
DE′	,,	,, AB	,,	,,	,, C

Because FD′ is antiparallel to CA
therefore BK bisects FD′
Similarly CK bisects DE′

Now $\angle D'DK = \angle A = \angle DD'K$;

therefore $KD = KD'$,

therefore $KD = KD' = KE' = KF$

Again $\angle E'EK = \angle B = \angle EE'K$;

therefore $KE = KE'$

Similarly $KF = KF'$;

[redacted]

Let BB′ and CC′ meet at K
Then BB′ is the locus of points whose distances from AB and BC are in the ratio $c : a$;
CC′ is the locus of points whose distances from AC and BC are in the ratio $b : a$;
therefore the ratio of the distances of K from AB and AC is $c : b$
that is, K lies on AA′

The eight varieties of position which the squares may occupy relatively to the sides of the triangle may be thus enumerated :

(1)	X outwardly	Y outwardly	Z outwardly
(2)	X inwardly	Y inwardly	Z inwardly
(3)	X inwardly	Y outwardly	Z outwardly
(4)	X outwardly	Y inwardly	Z inwardly
(5)	X outwardly	Y inwardly	Z outwardly
(6)	X inwardly	Y outwardly	Z inwardly
(7)	X inwardly	Y inwardly	Z outwardly
(8)	X outwardly	Y outwardly	Z inwardly

If the construction indicated in the enunciation of the third demonstration be carried out on these eight figures

(1)	and	(2)	will give the		insymmedian		point	K
(3)	„	(4)	„	„ „	first exsymmedian		„	K_1
(5)	„	(6)	„	„ „	second	„	„	K_2
(7)	„	(8)	„	„ „	third	„	„	K_3

Now that the existence of the insymmedian point is established, it may be well to give that property of the point which was the first to be discovered.

*The sum of the squares of the distances of the insymmedian point from the sides is a minimum**

* "Yanto" in Leybourn's *Mathematical Repository*, old series, III. 71 (1803). See *Proceedings of the Edinburgh Mathematical Society*, XI. 92-102 (1893)

In the identity

$$(x^2+y^2+z^2)(a^2+b^2+c^2)-(ax+by+cz)^2$$
$$=(bz-cy)^2+(cx-az)^2+(ay-bx)^2$$

let a b c denote the sides of the triangle,

x y z the distances of any point from the sides

Then the left side of the identity is a minimum when the right sides is a minimum

But $a^2+b^2+c^2$ is fixed, and so is $ax+by+cz$, since it is equal to $2\triangle$; therefore $x^2+y^2+z^2$ is a minimum when the right side is 0

Now the right side is the sum of three squares, and can only be 0 when each of the squares is 0;

therefore $$bz-cy=cx-az=ay-bx=0$$

therefore $$\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$$

Hence the point which has the sum of the squares of its distances from the sides a minimum is that point whose distances from the sides are proportional to the sides

[The proof here given is virtually that of Mr Lemoine, in his paper communicated to the Lyons meeting (1873) of the *Association Française pour l'avancement des Sciences*. Another demonstration by Professor Neuberg will be found in Rouché et de Comberousse's *Traité de Géométrie*, First Part, p 435 (1891)]

[redacted]

Then $$BR : CR = AB^2 : AC^2$$
$$C S' : AS' = BC^2 : BA^2$$
$$AT' : B T' = CA^2 : CB^2$$

therefore $$\frac{BR}{CR} \cdot \frac{CS'}{AS'} \cdot \frac{AT'}{BT'} = -1$$

since of the ratios $BR : CR$, $CS' : AS'$, $AT' : BT'$ two are positive and one negative ;
therefore AR BS′ CT′ are concurrent

Hence also AR′ BS CT′ ; AR′ BS′ CT are concurrent

The points of concurrency of

AR BS′ CT′ ; AR′ BS CT′ ; AR′ BS′ CT

will be called the 1st 2nd 3rd exsymmedian points, and will be denoted by K_1 K_2 K_3

Second Demonstration*

Figure 22

About ABC circumscribe a circle ; draw BK_1 CK_1 tangents to it at B C

Then BK_1 CK_1 are the exsymmedians from B C : to prove AK_1 to be the insymmedian from A

Through K_1 draw DE antiparallel to BC, and let AB AC meet it at D E

Then $$\angle BDK_1 = \angle ACB = \angle DBK_1 ;$$
therefore $$BK_1 = DK_1$$
Similarly $$CK_1 = EK_1 ;$$
therefore $$DK_1 = EK_1 ;$$
therefore AK_1 is the insymmedian from A

From this mode of demonstration it is clear that if K_1 be taken as centre and K_1B or K_1C as radius and a circle be described, that circle will cut AB AC at the extremities of a diameter

* Professor J. Neuberg in *Mathesis*, I. 173 (1881)

(1) *The insymmedians of a triangle pass through the poles of the sides of the triangle with respect to the circumcircle*

For K_1 is the pole of BC with respect to the circumcircle

(2) The six internal and external symmedians of a triangle meet three and three in four points which are collinear in pairs with the vertices

FIGURE 25

(3) If triangle ABC be acute-angled, the points

A	B	C	will be situated on the lines
K_2K_3	K_3K_1	K_1K_2;	

and the circle ABC will be the incircle of triangle $K_1K_2K_3$

If, however, triangle ABC be obtuse-angled, suppose at C,

then the point A will be situated on K_2K_3 produced

B " " " " K_3K_1 "

C " " " " K_1K_2;

and the circle ABC will be an excircle of triangle $K_1K_2K_3$

FIGURE 26

(4) Hence the relation in which triangle ABC stands to $K_1K_2K_3$ will, if ABC be acute-angled, be that in which triangle DEF stands to ABC; or, if ABC be obtuse-angled, it will be that in which one

[redacted]

FIGURE 27

$$\left.\begin{array}{lll} AD & BE & CF \\ AD_1 & BE_1 & CF_1 \\ AD_2 & BE_2 & CF_2 \\ AD_3 & BE_3 & CF_3 \end{array}\right\}\text{ are }\left\{\begin{array}{l} \Gamma \\ \Gamma_1 \\ \Gamma_2 \\ \Gamma_3 \end{array}\right.$$

Γ being called* the Gergonne point of ABC, and Γ_1 Γ_2 Γ_3 the associated Gergonne points

Hence the Gergonne point and its associates are the insymmedian points of the four DEF triangles

(7) With respect to BC
D and D_1 are isotomic points, so are D_2 and D_3; and a similar relation holds for the E points with respect to CA, and for the F points with respect to AB. Hence the triads

$$\left.\begin{array}{lll} AD_1 & BE_2 & CF_3 \\ AD & BE_3 & CF_2 \\ AD_3 & BE & CF_1 \\ AD_2 & BE_1 & CF \end{array}\right\}\text{ which are concurrent† at }\left\{\begin{array}{l} J \\ J_1 \\ J_2 \\ J_3 \end{array}\right.$$

furnish the four pairs of reciprocal points,

$$\begin{array}{lllll} \Gamma & \Gamma_1 & \Gamma_2 & \Gamma_3 & \text{(Gergonne points)} \\ J & J_1 & J_2 & J_3 & \text{(Nagel points)} \end{array}$$

(8) Since AD passes through J_1
BE „ „ J_2
CF „ „ J_3
therefore Γ is situated on each of the straight lines AJ_1 BJ_2 CJ_3; in other words, the triangles ABC $J_1J_2J_3$ are homologous and have Γ for centre of homology

* By Professor J. Neuberg. J. D. Gergonne (1771-1859) was editor of the *Annales de Mathématiques* from 1810 to 1831

† Many of the properties of the J points were given by C. H. Nagel in his *Untersuchungen über die wichtigsten zum Dreiecke gehörigen Kreise* (1836). This pamphlet I have never been able to procure. Since 1836 some of these properties have been rediscovered several times

Since AD_1 passes through Γ_1

BE_2 „ „ Γ_2

CF_3 „ „ Γ_3;

therefore J is situated on each of the straight lines $A\Gamma_1$ $B\Gamma_2$ $C\Gamma_3$; in other words, the triangles ABC $\Gamma_1\Gamma_2\Gamma_3$ are homologous and have J for centre of homology

Similarly ABC is homologous with

$$J\,J_3J_2 \qquad J_3J\,J_1 \qquad J_2J_1J$$

the centres of homology being respectively

$$\Gamma_1 \qquad \Gamma_2 \qquad \Gamma_3;$$

and ABC is homologous with

$$\Gamma\,\Gamma_3\Gamma_2 \qquad \Gamma_3\Gamma\,\Gamma_1 \qquad \Gamma_2\Gamma_1\Gamma$$

the centres of homology being respectively

$$J_1 \qquad J_2 \qquad J_3$$

(9) *If* $\Gamma' \qquad \Gamma_1' \qquad \Gamma_2' \qquad \Gamma_3'$

be the points of concurrency of lines drawn from A′ B′ C′, the mid points of the sides, parallel to the triads of angular transversals which determine the points

$$\Gamma \qquad \Gamma_1 \qquad \Gamma_2 \qquad \Gamma_3$$

then $\Gamma\Gamma' \qquad \Gamma_1\Gamma_1' \qquad \Gamma_2\Gamma_2' \qquad \Gamma_3\Gamma_3'$

[redacted]

(11) The J points are anticomplementary to the I points, and the tetrad*

$$I\,J \qquad I_1J_1 \qquad I_2J_2 \qquad I_3J_3$$

are concurrent at G the centroid of ABC

(12) J_1 J_2 J_3 J form an orthic tetrastigm *

FIGURE 26

(13) A I B I C I intersect E F F D D E

at the feet of the medians of triangle DEF ;

AD BE CF intersect E F F D D E

at the feet of the internal symmedians

AI_1 BI_1 CI_1 intersect E_1F_1 F_1D_1 D_1E_1

at the feet of the medians of triangle $D_1E_1F_1$;

AD_1 BE_1 CF_1 intersect E_1F_1 F_1D_1 D_1E_1

at the feet of the internal symmedians

Similarly for triangles $D_2E_2F_2$ $D_3E_3F_3$

(14) *The external symmedians of any triangle are also the external symmedians of three other associated triangles*

FIGURE 26

Let DEF be the triangle

Circumscribe a circle about DEF, and draw tangents to it at D E F. Let these tangents intersect at A B C. Then $D_1E_1F_1$ $D_2E_2F_2$ $D_3E_3F_3$ are the three triangles associated with DEF

To determine their vertices it is not necessary to find I_1 I_2 I_3 and to draw perpendiculars to BC CA AB

Make $CD_1 = BD$ $CE_1 = CD_1$ and $BF_1 = BD_1$ and the triangle $D_1E_1F_1$ is determined

Similarly for $D_2E_2F_2$ $D_3E_3F_3$

---

* William Godward in *Mathematical Questions from the Educational Times* II. 87, 88 (1865)

(15) *I_1A' I_2B' I_3C' are concurrent* at the insymmedian point of $I_1I_2I_3$*

FIGURE 26

For BC is antiparallel to I_2I_3 with respect to $\angle I_2I_1I_3$ and A′ is its mid point ;

therefore I_1A' is the insymmedian of $I_1I_2I_3$ from I_1

Similarly I_2B' „ „ „ „ „ „ I_2

and I_3C' „ „ „ „ „ „ I_3

(16) $$\left.\begin{array}{lll} I\,A' & I_3B' & I_2C' \\ I_3A' & I\,B' & I_1C' \\ I_2A' & I_1B' & I\,C' \end{array}\right\}\text{ are concurrent †}$$

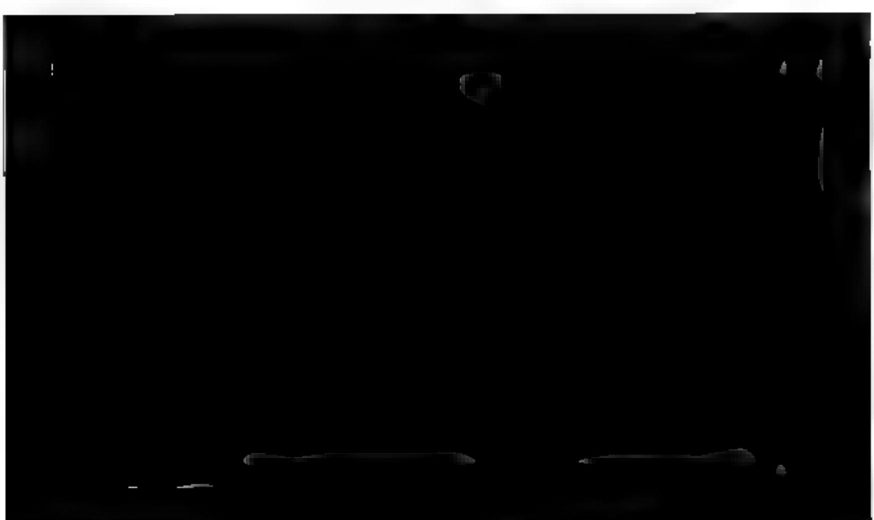
respectively at the insymmedian points of the triangles

$I\,I_3I_2$ $I_3I\,I_1$ I_2I_1I

§ 5

The internal and external symmedians from any vertex are conjugate harmonic rays with respect to the sides of the triangle which meet at that vertex ‡

FIGURE 25



Hence also for C S A S′ and A T B T′

(1) The following triads of points are collinear :

R′ S T; R S′ T; R S T′; R′ S′ T′

(2) *The following are harmonic ranges* *

$A\ K\ R\ K_1$; $B\ K\ S\ K_2$; $C\ K\ T\ K_3$

$A\ K_2\ R'\ K_2$; $B\ K_1\ S'\ K_3$; $C\ K_1\ T'\ K_2$

For B R C R′ is a harmonic range ;
therefore A.BRCR′ is a harmonic pencil ;
and its rays are cut by the transversals $BKSK_2$ and $B\ K_1\ S'\ K_3$;
therefore $B\ K\ S\ K_2$ $B\ K_1\ S'\ K_3$ are harmonic ranges

(3) *If D E F be the points in which AK BK CK cut the circumcircle of ABC, then the following are harmonic ranges*

$A\ R\ D\ K_1$; $B\ S\ E\ K_2$; $C\ T\ F\ K_3$

FIGURE 25

For K_1B K_1C are tangents to the circle ABC, and K_1DRA is a secant through K_1 ;
therefore this secant is cut harmonically † by the chord of contact BC and the circumference

(4) *R′ is the pole of AK, with respect to the circumcircle* ‡

Since	AR′ is the tangent at A
therefore	AR′ is the polar of A
Now	BR′ „ „ „ „ K_1 ;
therefore	R′ is the pole of AK_1

Similarly for S′ and T′

(5) *R′D S′E T′F are tangents to the circumcircle*

For AD is the polar of R′ with respect to the circumcircle, that is, AD is the chord of contact of the tangents from R′

* The first of these is mentioned by Dr Franz Wetzig in Schlömilch's *Zeitschrift*, XII. 289 (1867)

† This is one of Apollonius's theorems. See his *Conics*, Book III., Prop. 37-40

‡ C. Adams's *Eigenschaften des...Dreiecks*, pp. 3-4 (1846)

(6) *The straight line $R'S'T'$ is the polar of K with respect to the circumcircle*

For AK_1 BK_2 CK_3 pass through K ;
therefore their respective poles R′ S′ T′ will lie on the polar of K

(7) R′S′T′ is perpendicular to OK, and its distance from O is equal to R^2/OK, where R denotes the radius of the circumcircle

R′S′T′ is sometimes called *Lemoine's line*

(8) *$R'S'T'$ is the trilinear polar* of K, or it is the line harmonically associated with the point K*

For	ST TR RS meet BC CA AB
at	R′ S′ T′ ;
therefore	R′S′T′ is the trilinear polar of K

(9) The three triangles ABC RST $K_1K_2K_3$ taken in pairs will have the same axis of homology, namely the trilinear polar of K

(10) *The following triads of points are collinear*

$$R'\ E\ F; \quad S'\ F\ D; \quad T'\ D\ E$$

For BCEF is an encyclic quadrilateral,
and BE CF intersect at K ;
therefore EF intersects BC on the polar of K
Now the polar of K intersects BC at R′ ;

Again BCEF is an encyclic quadrilateral, and

BE CF intersect at K

BC EF „ „ R′

BF CE „ „ D′;

therefore triangle KR′D′ is self-conjugate with respect to the circumcircle;

therefore KD′ is the polar of R′

But AK „ „ „ „ R′;

therefore A D D′ are collinear

(12) The following triads of lines are concurrent:

	AK BF CE;	BK CD AF;	CK AE BD
at	D′;	E′;	F′

and D′ E′ F′ are situated on R′S′T′

(13) *The straight lines which join the mid point of each side of a triangle to the mid point of the corresponding perpendicular of the triangle are concurrent at the insymmedian point* *

FIGURE 23

Let K K_1 be the insymmedian and first exsymmedian points of ABC;

let A′ be the mid point of BC and let A′K meet the perpendicular AX at P_1

Join A′K_1

Then A′K_1 is parallel to AX

Now since A K R K_1 is a harmonic range

therefore A′.AKRK_1 is a harmonic pencil;

therefore AX which is parallel to the ray A′K_1 is bisected by the conjugate ray A′K

§ 6

If L M N be the projections of K on the sides, then K is the centroid † *of triangle LMN*

* Dr Franz Wetzig in Schlömilch's *Zeitschrift*, XII. 289 (1867)

† E. W. Grebe in Grunert's *Archiv*, IX. 253 (1847)

FIGURE 28

Through L draw a parallel to MK, meeting NK produced in K′ Join K′M

Then triangle KLK′ has its sides respectively perpendicular to BC CA AB;

therefore KL : LK′ = BC : CA

But KL : KM = BC : CA

therefore LK′ = KM;

therefore KLK′M is a parallelogram;

therefore KK′ bisects LM,

that is, KN is a median of LMN

Similarly KL KM are medians;

therefore K is the centroid of LMN

Another demonstration by Professor Neuberg will be found in *Mathesis*, I. 173 (1881)

(1) *The sides of LMN are proportional to the medians of ABC, and the angles of LMN are equal to the angles which the medians of ABC make with each other* *

Since KL KM KN are two-thirds of the respective medians of LMN, and are proportional to BC CA AB;

therefore the medians of LMN are proportional to BC CA AB;

therefore the sides of LMN are proportional to the medians of ABC

[redacted]

(2) If LMN be considered as the fundamental triangle, K its centroid, and if at the vertices L M N perpendiculars be drawn to the medians KL KM KN a new triangle ABC is formed having K for its insymmedian point

(3) *The sum of the squares of the sides of the triangle LMN inscribed in ABC is less than the sum of the squares of the sides of any other inscribed triangle* *

The proof of this statement depends on the following lemma :

Given two fixed points M N and a fixed straight line BC ; that point L on BC for which $NL^2 + LM^2$ is a minimum is the projection on BC of the mid point of MN

(4) *If through every two vertices and the centroid of a triangle circles be described, the triangle formed by joining their centres will have for centroid and insymmedian point the circumcentre and the centroid of the fundamental triangle* †

FIGURE 39

Let O_1 O_2 O_3 be the centres of the three circles, and let the circles O_2 O_3 cut BC in the points E F

Join AG cutting BC in its mid point A′, and draw O_2P O_3Q perpendicular to BC

Then $\qquad A'B \cdot A'F = A'A \cdot A'G = A'C \cdot A'E$

therefore $\qquad A'F = A'E$

and $\qquad A'Q = A'P$

Now O_1A' bisects BC perpendicularly ;
therefore O_1A' passes through the circumcentre of ABC and bisects O_2O_3
therefore O_1A' is a median of triangle $O_1O_2O_3$

Similarly for O_2B' and O_3C' ;
therefore O the circumcentre of ABC is the centroid of $O_1O_2O_3$

---

* Mr Emile Lemoine in the *Journal de Mathématiques Élémentaires*, 2nd series, III. 52-3 (1884)

† This theorem and the proof of it have been taken from Professor W. Fuhrmann's *Synthetische Beweise planimetrischer Sätze*, pp. 101-2 (1890)

Again if through A B C perpendiculars be drawn to the medians AG BG CG these perpendiculars will form a triangle UVW whose vertices will be situated on the circumferences* of O_1 O_2 O_3 and which will be similar to the triangle $O_1O_2O_3$
Also the triangles UVW $O_1O_2O_3$ have G for their centre of similitude

**Now triangle UVW has G for its insymmedian point ;
therefore G is also the insymmedian point of triangle $O_1O_2O_3$**

(5) *If* L_1 M_1 N_1 L_2 M_2 N_2 L_3 M_3 N_3

be the projections on BC CA AB of

K_1 K_2 K_3

then $K_1N_1L_1M_1$ $K_2L_2M_2N_2$ $K_3M_3N_3L_3$

are parallelograms †

FIGURE 40

The points K_1 L_1 B N_1 are concyclic

therefore $\angle L_1K_1N_1 = \angle ABC$

The points K_1 M_1 C L_1 are concyclic

therefore $\angle K_1L_1M_1 = \angle K_1CM_1$

[redacted]

§ 7

If AK BK CK be produced to meet the circumcircle in D E F the triangle DEF has the same insymmedians as ABC

FIRST DEMONSTRATION

FIGURE 29

From K draw KL KM KN perpendicular to BC CA AB and join MN NL LM

Since the points B L K N are concyclic

therefore $\angle KLN = \angle KBN$
$= \angle EBA$
$= \angle EDA$

Since the points C L K M are concyclic

therefore $\angle KLM = \angle KCM$
$= \angle FCA$
$= \angle FDA$

Hence $\angle MLN = \angle EDF$

Similarly $\angle LMN = \angle DEF$ or $\angle MNL = \angle EFD$

and triangles LMN DEF are directly similar

But since $\angle KLN = \angle KDE$

and $\angle KLM = \angle KDF$

therefore the point K in triangle LMN corresponds to its isogonally conjugate point in triangle DEF

Now K is the centroid of triangle LMN ;

therefore K is the insymmedian point of triangle DEF

SECOND DEMONSTRATION

FIGURE 25

Let AR′ BS′ CT′ be the exsymmedians

Since AK is the polar of R′, and BC EF both pass through R′ not only will the tangents to the circumcircle at B C meet on the polar of R′ but also the tangents at E F

But the tangents at E F meet on the insymmedian of DEF from D; therefore the insymmedian AD is common to triangles ABC DEF

Similarly for the insymmedians BE CF

The cosymmedian triangles ABC DEF are homologous, the insymmedian point K being their centre of homology, and R′S′T′ their axis of homology

(1) *If two triangles be cosymmedian the sides of the one are proportional to the medians of the other* *

For triangle DEF is similar to triangle LMN

Or thus:

Let G be the centroid of ABC

Join GB GC

Then $\angle EDF = \angle EDA + \angle ADF$

$= \angle KBA + \angle KCA$

$= \angle GBC + \angle GCB$

since the points G K are isogonally conjugate

Similarly $\angle DEF = \angle GCA + \angle GAC$

and $\angle EFD = \angle GAB + \angle GAB$

A reference to the *Proceedings of the Edinburgh Mathematical Society*, I. 25 (1894) will show that this proves the theorem.

[redacted]

Hence $$\mathrm{DEF} = \frac{3\Delta \mathrm{R}^2}{4\mathrm{R}'^2}$$

$$= \frac{3\Delta}{4} \cdot \frac{a^2b^2c^2}{16\Delta^2} \cdot \left(\frac{3\Delta}{m_1m_2m_3}\right)^2$$

$$= \frac{27\Delta a^2b^2c^2}{64m_1^2m_2^2m_3^2}$$

The values of m_1 m_2 m_3 are given in the *Proceedings of the Edinburgh Mathematical Society*, I. 29 (1894)

(3) *If BD CD be joined, DR DR′ are an insymmedian and an exsymmedian of triangle* DCB*

FIGURE 24

Draw $\mathrm{AA_1}$ parallel to BC to meet the circumcircle at $\mathrm{A_1}$ and let $\mathrm{A_1K_1}$ meet the circle at $\mathrm{D_1}$

Then triangle $\mathrm{ACK_1}$ is congruent to $\mathrm{A_1BK_1}$

therefore $$\angle \mathrm{CAD} = \angle \mathrm{BA_1D_1}$$

therefore $\mathrm{DD_1}$ is parallel to BC

Now since BC is the polar of $\mathrm{K_1}$ and $\mathrm{AA_1}$ $\mathrm{DD_1}$ are parallel, therefore $\mathrm{AD_1}$ $\mathrm{A_1D}$ intersect on BC at its mid point A′

Again $$\angle \mathrm{CDR} = \angle \mathrm{CDA}$$
$$= \angle \mathrm{BDA_1}$$
$$= \angle \mathrm{BDA'}$$

therefore DR is isogonal to the median DA′
But DR′ is a tangent to the circumcircle at D;
therefore DR′ is an insymmedian

(4) Hence BR $\mathrm{BK_1}$ are an insymmedian and an exsymmedian of triangle BDA;
CR $\mathrm{CK_1}$ an insymmedian and an exsymmedian of triangle CAD

(5) $$AR'^2 + BK_1^2 = K_1R'^2$$

Let O be the centre of the circumcircle ABC

* C. Adams in his *Eigenschaften des...Dreiecks*, pp. 4-5 (1846) gives (3)—(7)

Then $$\begin{aligned} AR'^2 &= OR'^2 - OA^2 \\ &= OA'^2 + A'R'^2 - OA^2 \\ BK_1^2 &= A'B^2 + A'K_1^2 \\ &= OB^2 - OA'^2 + A'K_1^2 \end{aligned}$$

therefore $$\begin{aligned} AR'^2 + BK_1^2 &= A'R'^2 + A'K_1^2 \\ &= K_1R'^2 \end{aligned}$$

(6) *OR is perpendicular to K_1R'*

For R' is the pole of AK_1

and K_1 „ „ „ „ BC

therefore K_1R' is the polar of R

therefore OR is perpendicular to K_1R'

(7) *AR' is a mean proportional between $A'R'$ and RR'*

Since B R C R' form a harmonic range,
and A' is the mid point of BC

therefore $$BR' : A'R' = RR' : CR'$$

therefore $$\begin{aligned} A'R' \cdot RR' &= BR' \cdot CR' \\ &= AR'^2 \end{aligned}$$

(8) $$AB \cdot CD = AC \cdot BD = \tfrac{1}{2} AD \cdot BC$$

Figure 24

This name was suggested to Mr Tucker by Professor Neuberg in 1885

The first systematic study of harmonic quadrilaterals was made by Mr Tucker. In his article "Some properties of a quadrilateral in a circle, the rectangles under whose opposite sides are equal," read to the London Mathematical Society on 12th February 1885, he states that in his attempt to extend the properties of the Brocard points and circle to the quadrilateral he "was brought to a stand at the outset by the fact that the equality of angles does not involve the similarity of the figures for figures of a higher order than the triangle. Limiting the figures, however, by the restriction that they shall be circumscriptible" he arrived at a large number of beautiful results all of which cannot unfortunately be given here.

Starting with the encyclic quadrilateral ABCD whose diagonals intersect at E, and investigating the condition that a point P can be found such that

$$\angle PAB = \angle PBC = \angle PCD = \angle PDA$$

he finds, by analytical considerations, that a condition for the existence of such a point is that the rectangles under the opposite sides of the quadrilateral must be equal. He then shows that if there be one Brocard point P for the quadrilateral there will be a second P′; that the lines

$$PA \quad PB \quad PC \quad PD; \qquad P'A \quad P'B \quad P'C \quad P'D$$

intersect again in four points which, with P P′ lie on the circumference of a circle with diameter OE, where O is the centre of the circle ABCD.

Next, if through E parallels be drawn to the sides of the quadrilateral, these parallels will meet the sides in eight points which lie on a circle concentric with the previous one.

Lastly he shows that the symmedian points (ρ_1 ρ_2) of ABD BCD lie on AC; the symmedian points (σ_1 σ_2) of ABC ADC lie on BD; the lines $O\rho_1$ $O\rho_2$ $O\sigma_1$ $O\sigma_2$ are the diameters of the Brocard circles of the triangles ABD BCD ABC ACD respectively; the centres of the four Brocard circles lie two and two on straight lines, parallel to AC BD; the circles themselves intersect two and two on the diagonals AC BD at their mid points, that is, where the Brocard circle of the quadrilateral meets the diagonals.

Mr Tucker's researches were taken up by Messrs Neuberg and

Tarry, Dr Casey, and the Rev. T. C. Simmons, and there now exists a tolerably extensive theory of harmonic polygons. The reader who wishes to pursue this subject may consult

Mr R. Tucker's memoir which appeared in *Mathematical Questions from the Educational Times*, Vol. XLIV. pp. 125–135 (1886)

Professor Neuberg *Sur le Quadrilatère Harmonique* in *Mathesis*, V. 202–204, 217–221, 241–248, 265–269 (1885)

Dr John Casey's memoir (read 26th January 1886) "On the harmonic hexagon of a triangle" in the *Proceedings of the Royal Irish Academy*, 2nd series, Vol. IV. pp. 545–556

A memoir by Messrs Gaston Tarry and J. Neuberg *Sur les Polygones et les Polyèdres Harmoniques* read at the Nancy meeting (1886) of the *Association Française pour l'avancement des sciences*. See the Report of this meeting, second part, pp. 12–26

A memoir by the Rev. T. C. Simmons (read 7th April 1887) "A new method for the investigation of Harmonic Polygons" in the *Proceedings of the London Mathematical Society*, Vol. XVIII. pp. 289–304

Dr Casey's *Sequel to the First Six Books of the Elements of Euclid*, 6th edition, pp. 220–238 (1892)

§ 8

The Cosine or second Lemoine Circle

If through the insymmedian point of a triangle, antiparallels be [redacted]

[This theorem was first given by Mr Lemoine at the Lyons meeting (1873) of the *Association Française pour l'avancement des sciences*, and the circle determined by it has hence been called one of Lemoine's circles (the second).

The existence of the circle however, and the six points through which it passes were discovered by Mr Stephen Watson of Haydonbridge in 1865, and its diameter expressed in terms of the sides of the triangle. See *Lady's and Gentleman's Diary* for 1865, p. 89, and for 1866, p. 55

In the same publication Mr Thomas Milbourn in 1867 announced a neat relation connecting the diameter of this circle with the diameter of the circumcircle, and here, as far as the *Diary* is concerned, the inquiry seemed to have stopped]

(1) The figures DD'E'F EE'F'D FF'D'E are rectangles *

It may be interesting to give the way in which these three rectangles made their first appearance.

(2) *Three rectangles may be inscribed in any triangle so that they may have each a side coincident in direction with the respective sides of the triangle, and their diagonals all intersecting in the same point, and one circle may be circumscribed about all the three rectangles* †

FIGURE 30

Let ABC be the triangle

Draw AX perpendicular to BC ;
and produce CB to Q making BQ equal to CX
About ACQ circumscribe a circle cutting AB at P
Join PC ; and draw BE parallel to PC and meeting AC at E
From E draw ED' parallel to AB and EF perpendicular to AB, and let these lines meet BC AB at D' F
About D'EF circumscribe a circle cutting BC CA AB again in D E' F'. The six points D D' E E' F F' are the vertices of the required rectangles

Draw CZ perpendicular to AB,
and let ED' meet CP at R

The similar triangles ABX CBZ give

$$AX : CZ = AB : CB$$
$$= BQ : BP$$
$$= CX : BP;$$

therefore
$$AX : CX = CZ : BP$$

---

* Mr Lemoine at the Lille meeting (1874) of the *Association Française pour l'avancement des sciences*

† Mr Stephen Watson in the *Lady's and Gentleman's Diary* for 1865, p. 89, and for 1866, p. 55

But $\quad EF : CZ = AE : AC$

$\qquad = AB : AP$

$\qquad = ED' : ER$

$\qquad = ED' : BP$;

therefore $\quad EF : ED' = CZ : BP$;

therefore $\quad AX : CX = EF : ED'$;

therefore the right-angled triangles AXC FED′ are similar

Hence $\quad \angle XAC = \angle EFD' = \angle D'E'E$;

therefore $\quad D'E'$ is parallel to AX

Now $\quad \angle FE'D' = \angle FED' =$ a right angle ;

therefore FE′ is parallel to BC,
and DD′E′F is one of the rectangles

Again because $\angle EFF'$ is right,
therefore EF′ is a diameter ;
therefore $\angle F'D'E$ is right, as well as $\angle F'DE$ $\angle F'E'E$;
therefore EE′F′D FF′D′E are the other rectangles

(3) *To find the diameter* of the circle DEF*

FIGURE 30

$$AB^2 \quad = BC^2 + CA^2 - 2BC \cdot CX$$

[redacted]

Hence $\dfrac{2\triangle}{\mathrm{EF}} = \dfrac{a^2+b^2+c^2}{2c}$

and $\mathrm{EF} = \dfrac{4c\triangle}{a^2+b^2+c^2}$

Lastly $\mathrm{D'F} : \mathrm{EF} = \mathrm{AC} : \mathrm{AX}$

therefore $\mathrm{D'F} : \dfrac{4c\triangle}{a^2+b^2+c^2} = b : \dfrac{2\triangle}{a}$

therefore $\mathrm{D'F} = \dfrac{2abc}{a^2+b^2+c^2}$ = the diameter

The following is another proof

FIGURE 19

Triangles AEF′ ABC are similar
and AK is a median of AEF′;

therefore $\mathrm{EF'} : \mathrm{AK} = \mathrm{BC} : m_1$

therefore $\mathrm{EF'} = \dfrac{\mathrm{AK} \cdot \mathrm{BC}}{m_1}$

$= \dfrac{2abc}{a^2+b^2+c^2}$

For the value of AK, namely, $\dfrac{2abm_1}{a^2+b^2+c^2}$

see Formulæ connected with the Symmedians, at the end of this paper.

(4) *If d denote the diameter of circle DEF,*
and D ,, ,, ,, ,, ,, ABC,
*then** $\dfrac{1}{d^2}+\dfrac{1}{D^2}=\dfrac{1}{a^2}+\dfrac{1}{b^2}+\dfrac{1}{c^2}$

For $\dfrac{1}{d}=\dfrac{a^2+b^2+c^2}{2abc} \qquad \dfrac{1}{\mathrm{D}}=\dfrac{4\triangle}{2abc}$;

therefore $\dfrac{1}{d^2}+\dfrac{1}{\mathrm{D}^2}=\dfrac{(a^2+b^2+c^2)^2+(4\triangle)^2}{4a^2b^2c^2}$

---

* Mr Thomas Millbourn in the *Lady's and Gentleman's Diary* for 1867, p. 71, and for 1868, p. 75

$$=\frac{(a^2+b^2+c^2)^2+4c^2a^2-(a^2-b^2+c^2)^2}{4a^2b^2c^2}$$

$$=\frac{4c^2a^2+2(c^2+a^2)2b^2}{4a^2b^2c^2}$$

$$=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$$

(5) *The centre of the circle DEF is the insymmedian point of the triangle ABC*

FIGURE 30

Because $\angle EFD' = \angle XAC$

therefore their complements are equal

that is $\angle D'FB = \angle ACX$;

therefore D'F is antiparallel to CA with respect to B

Hence E'D " " " AB " " " C

and F'E " " " BC " " " A

But these antiparallels are all bisected at the centre of the circle DEF;

therefore the centre of the circle is the insymmedian point K

[redacted]

FIGURE 19

For $\angle DEF = \angle DD'F$

$= \angle A$

and $\angle EFD = \angle EE'D$

$= \angle B$

therefore EFD is similar to ABC

In like manner for F'D'E'

Now since EFD F'D'E' are similar to each other and are inscribed in the same circle they are congruent

(8) The angles which	F D	D E	E F
make with	A B	B C	C A
are equal to the angles which	F'D'	D'E'	E'F'
make with	B C	C A	A B

§ 9

THE TRIPLICATE RATIO OR FIRST LEMOINE CIRCLE

If through the insymmedian point of a triangle parallels be drawn to the three sides, the six points in which they meet the sides will be concyclic

FIRST DEMONSTRATION *

FIGURE 31

Let K be the insymmedian point of ABC,

and through K let there be drawn EF' FD' DE'

respectively parallel to BC CA AB,

the D points being on BC, the E's on CA, the F's on AB

Then AFKE' is a parallelogram;

therefore AK bisects FE';

therefore F E' is antiparallel to BC ;

therefore F E' „ „ „ EF';

* This mode of proof is due to Mr R. F. Davis. See Fourteenth General Report (1888) of the Association for the Improvement of Geometrical Teaching, p. 39.

therefore the points E E′ F F′ are concyclic
Hence „ „ F F′ D D′ „ „
and „ „ D D′ E E′ „ „

Now if these three circles be not one and the same circle, their radical axes, which are DD′ EE′ FF′ or BC CA AB, must meet in a point, the radical centre
But BC CA AB do not meet in a point;
therefore these three circles are one and the same,
that is, the six points D D′ E E′ F F′ are concyclic

SECOND DEMONSTRATION

FIGURE 32

Let O be the circumcentre of ABC, and let OA OK be joined.

Because AFKE′ is a parallelogram
therefore AK bisects FE′ at U;
therefore FE′ is antiparallel to BC
therefore OA is perpendicular to FE′
If therefore through U a perpendicular be drawn to FE′
it will be parallel to OA, and will pass through O′,
the mid point of OK

Hence also the perpendiculars to DF′ and ED′ through their mid points V and W will pass through O′;
that is, O′ the mid point of OK is the centre of a circle which passes

[redacted]

The parallels drawn through K, the symmedian point, to the sides of ABC are often called Lemoine's parallels, and the hexagon they determine DD′EE′FF′ Lemoine's hexagon.

(1) *If E′F F′D D′E be produced to meet and form a triangle, then the incircle of this triangle will have O′ for its centre, and its radius will be half the radius of the circumcircle of ABC*

For $\quad O'U = \frac{1}{2}OA \quad O'V = \frac{1}{2}OB \quad O'W = \frac{1}{2}OC$;

therefore $\quad O'U = O'V = O'W$;

and O′U is perpendicular to E′F, O′V to F′D, O′W to D′E

(2) The figures DD′EF′ EE′FD′ FF′DE′ are symmetrical trapeziums;

therefore $\quad E'F = F'D = D'E$

(3) *Triangles FDE E′F′D′ are directly similar to ABC and congruent to each other*

For $\quad \angle FDE = \angle FF'E = \angle B$

and $\quad \angle DEF = \angle DD'F = \angle C$

therefore FDE is similar to ABC

In like manner for E′F′D′

Now since FDE E′F′D′ are similar to each other and are inscribed in the same circle, therefore they are congruent.

It is not difficult to show that if K be any point in the plane of ABC and through it parallels be drawn to the sides, as in the figure, the triangles DEF D′E′F′ are equal in area.

See Vuibert's *Journal de Mathématiques Élémentaires*, VIII. 12 (1883)

(4) *The following three triangles are directly similar to ABC:*

KDD′ E′KE FF′K

For EF′ FD′ DE′ are parallel to the sides

(5) *The following six triangles are inversely similar to ABC:*

AE′F KFE′ DBF′ F′KD D′EC ED′K

For E′F F′D D′E are antiparallel to the sides

(6) *The triangles* cut off from ABC by E′F F′D D′E are together equal to triangle DEF or D′E′F′*

---

* Properties (6)—(9) are due to Mr Tucker. See *Quarterly Journal*, XIX. 344, 346 (1883)

For $\qquad AE'F = \frac{1}{2}AE'KF = EKF$

$$DBF' = \frac{1}{2}BF'KD = FKD$$

$$D'EC = \frac{1}{2}CD'KE = DKE$$

therefore $\qquad AE'F + DBF' + D'EC = DEF = D'E'F'$

(7) *The following six angles are equal :*

$$DFB \quad EDC \quad FEA \qquad D'E'C \quad F'D'B \quad E'F'A$$

For the arcs E'F F'D D'E are equal

(8) If each of these angles be denoted by ω

$$\angle AFE = \angle AE'F' = B + C - \omega$$

$$\angle BDF = \angle BF'D' = C + A - \omega$$

$$\angle CED = \angle CD'E' = A + B - \omega$$

(9) The following points are concyclic :

$$B \; C \; E' \; F \qquad C \; A \; F' \; D \qquad A \; B \; D' \; E$$

(10) *The radical axis of the circumcircle and the triplicate ratio circle is the Pascal line of Lemoine's hexagon*

Let FE' meet BC at X

Since the points B F E' C are concyclic,

therefore $\qquad XB \cdot XC = XF \cdot XE'$

[redacted]

FIGURE 33

(13) *If the chords*

$$EF\ E'F' \qquad FD\ F'D' \qquad DE\ D'E'$$

intersect in $\qquad p \qquad q \qquad r$

the triangles ABC pqr are homologous *

Let Bq Cr meet at T, and, for the moment, denote the distances of T from BC CA AB by α β γ

Then $\alpha : \gamma =$ perp. on BC from q : perp. on AB from q

$$= \text{DD}' : \text{FF}'$$

from the similarity of triangles DD'q F'Fq

$$= a^3 : c^3$$

Similarly $\alpha : \beta = a^3 : b^3$;

therefore $\beta : \gamma = b^3 : c^3$;

therefore the point p lies on AT

(14) *The intersections of the antiparallel chords with Lemoine's parallels, that is, of*

$$E'F\ EF'' \qquad F''D\ FD' \qquad D'E\ DE'$$

namely $\qquad P \qquad Q \qquad R$

are collinear *

The quadrilateral EE'FF' is inscribed in the circle DEF, and

EE' FF' $\qquad$ EF E'F' $\qquad$ E'F EF'

meet in $\qquad$ A $\qquad p \qquad$ P

therefore triangle ApP is self-conjugate with respect to circle DEF; therefore P is the pole of Ap with respect to DEF

Similarly Q „ „ „ Aq „ „ „; „

and R „ „ „ Ar „ „ „ „

Now Ap Bq Cr are concurrent at T

therefore P Q R are collinear on the polar of T with respect to the circle DEF

* Dr John Casey. See his *Sequel to Euclid*, 6th ed., p. 190 (1892)

(15) *If the intersections of*

	DE F'D'	*EF' D'E'*	*FD E'F'*
be	*l*	*m*	*n*

the triangles ABC lmn are similar and oppositely situated

Since the arcs E'F F'D D'E are equal

therefore $\angle E'mF = \angle E'nF = 2\omega$;

therefore the points E' m n F are concyclic;

therefore $\angle E'nm = \angle E'Fm = \angle E'F'E$;

therefore mn is parallel to EF' and to BC

Similarly for the other sides

(16) *The triangles pqr lmn are homologous and K is their centre of homology*

For E'F'D'FED is a Pascal hexagram;

therefore the intersections of

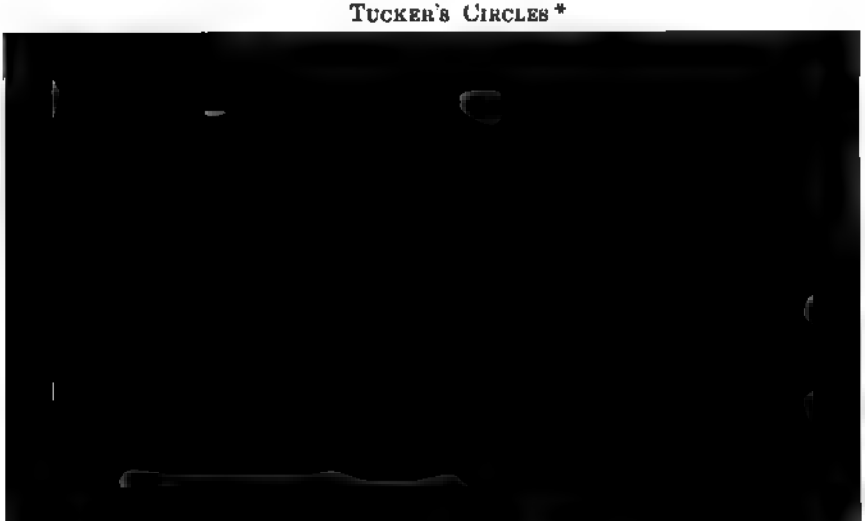
	E'F' FE	F'D' ED	D'F DE'
namely	p	l	K

are collinear

Similarly q m K; r n K are collinear

§ 10

Tucker's Circles *



therefore the points E E′ F F′ are concyclic

Similarly „ „ F F′ D D′ „ „

and „ „ D D′ E E′ „ „

therefore the six points D D′ E E′ F F′ are concyclic

(1) *To find the centre of the circle DD′EE′FF′* *

Let O O_1 be the circumcentres of ABC $A_1B_1C_1$

Then O O_1 K are collinear
and OA is parallel to O_1A_1
Now since E′F is antiparallel to BC
therefore OA is perpendicular to E′F
Hence if E′F meet AA_1 at U,
a line through U parallel to AO will bisect E′F perpendicularly, and also bisect OO_1
Similarly the perpendicular bisectors of F′D and D′E will bisect OO_1 ;
therefore the centre of the circle is the mid point of OO_1

(2) *Triangles FDE E′F′D′ are directly similar to ABC and congruent to each other*

FIGURE 34

Since F′E D′F E′D are respectively parallel
to BC CA AB
therefore the arcs E′F F′D D′E are equal ;

therefore $\angle$ EFD = $\angle$ D′FF′ = $\angle$ A

Similarly $\angle$ FDE = $\angle$ E′DD′ = $\angle$ B

therefore FDE is similar to ABC

In like manner for E′F′D′

Now since FDE E′F′D′ are similar to each other and are inscribed in the same circle therefore they are congruent

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* The properties (1), (2), (3), (6), (7) are due to Mr Tucker. See *Quarterly Journal*, XX. 59, 57, XIX. 348, XX. 59 (1884, 3)

(3) If T be the mid point of OO_1

$$TU = \tfrac{1}{2}(OA + O_1A_1)$$

Similarly, if V W be the mid points of F′D D′E,

$$TV = \tfrac{1}{2}(OB + O_1B_1)$$

$$TW = \tfrac{1}{2}(OC + O_1C_1)$$

Hence if E′F F′D D′E be produced to meet and form a triangle $A_2B_2C_2$

T will be the incentre of the triangle $A_2B_2C_2$ and the radius of the incircle will be an arithmetic mean between the radii of the circumcircles of ABC and $A_1B_1C_1$

(4) The triangle $A_2B_2C_2$ formed by producing E′F F′D D′E will have its sides respectively parallel to those of $K_1K_2K_3$ formed by drawing through A B C tangents to the circumcircle ABC

FIGURE 35

(5) Triangles $A_2B_2C_2$ $K_1K_2K_3$ have K for their centre of homology

(6) When the triangle $A_1B_1C_1$ becomes the triangle ABC, the Tucker circle DD′EE′FF′ becomes the circumcircle

[redacted]

FIGURE 28

Let R denote the foot of the symmedian AK

Then $KF' : KE = BD : CD'$

$$= c^2 : b^2$$

and $RD : RD' = BR - BD : CR - CD'$

$$= c^2 : b^2$$

therefore F′D KR ED′ are concurrent

§11

TAYLOR'S CIRCLE

The six projections of the vertices of the orthic triangle on the sides of the fundamental triangle are concyclic

FIGURE 36

Let the projections of X on CA AB be Y_1 Z_1

" " " Y " AB BC " Z_2 X_2

" " " Z " BC CA " X_3 Y_3

Then $AZ : AZ_1 = AH : AX$

$$= AY : AY_1$$

therefore YZ is parallel to Y_1Z_1

Now Y_3Z_2 is antiparallel to YZ

therefore Y_3Z_2 " " Y_1Z_1

therefore Y_1 Y_3 Z_2 Z_1 are concyclic

Similarly Z_2 Z_1 X_3 X_2 " "

and X_3 X_2 Y_1 Y_3 " "

therefore the six points X_3 X_2 Y_1 Y_3 Z_2 Z_1 are concyclic

The property, that the six projections of the vertices of the orthic triangle on the sides of the fundamental triangle are concyclic, seems to have been first published in Mr Vuibert's *Journal de Mathématiques Élémentaires* in November 1877. See Vol. II. pp. 30, 43. It is proposed by Eutaris. This name, as my friend Mr Maurice D'Ocagne informs me, was assumed anagrammatically by M. Restiau, at that time a répétiteur in the Collège Chaptal, Paris.

The same property, along with three others, is given in Catalan's *Théorèmes et Problèmes*, 6th ed., pp. 132-4 (1879). It occurs also in a question proposed by Professor Neuberg in *Mathesis*, I. 14 (1881), and in a paper by Mr H. M. Taylor in the *Messenger of Mathematics*, XI. 177-9 (1882). A proof by Mr C. M. Jessop, somewhat shorter than that given by Mr Taylor, occurs in the *Messenger*, XII. 36 (1883) and in the same volume (pp. 181-2) Mr Tucker examines whether any other positions of X Y Z on the sides would, with a similar construction, give a six-point circle, and he shows that no other circle is possible under the circumstances.

See also *L'Intermédiaire des Mathématiciens*, II. 166 (1895).

The projections of

X on BY CZ are Y_2 Z_0
Y „ CZ AX „ Z_2 X_0
Z „ AX BY „ X_1 Y_0

With regard to the notation it may be remarked that the
X points lie on BC and on the perpendicular to it from A
Y „ „ „ CA „ „ „ „ „ „ „ B
Z „ „ „ AB „ „ „ „ „ „ „ C

Let a notation, similar to that which prevails with regard to the sides, the semiperimeter, the radii of the incircle and the excircles of triangle ABC, be adopted for triangle XYZ; that is, let

$$YZ = x \qquad ZX = y \qquad XY = z$$

$\sigma = \frac{1}{2}(x+y+z)$ $\sigma_1 = \frac{1}{2}(-x+y+z)$ $\sigma_2 = \frac{1}{2}(x-y+z)$ $\sigma_3 = \frac{1}{2}(x+y-z)$

and let ρ ρ_1 ρ_2 ρ_3 be the radii of the incircle and the excircles

[redacted]

(2) $\quad Y_1Z_1 \quad Z_2X_2 \quad X_3Y_3$

or $\quad Y_2Z_0 \quad Z_3X_0 \quad X_1Y_0$

intersect two by two at the mid points of the sides of XYZ

(3) *

$$Y_1Z_1 = Z_2X_2 = X_3Y_3 = \sigma$$
$$Y_2Z_0 = Z_3X_2 = X_3Y_0 = \sigma_1$$
$$Y_1Z_0 = Z_3X_0 = X_1Y_3 = \sigma_2$$
$$Y_2Z_1 = Z_2X_0 = X_1Y_0 = \sigma_3$$

(4) If $X' \quad Y' \quad Z'$ be the mid points of
YZ ZX XY
the sides of triangle X'Y'Z' intersect the sides of ABC in six concyclic points

(5) *Triangles ABC X'Y'Z' are homologous, and the symmedian point K is the centre of homology*

For YZ is antiparallel to BC,
and X' is the mid point of YZ;
therefore AX' is the symmedian from A
Similarly BY' CZ' are the symmedians from B C

(6) †

$$R \cdot Y_1Z_1 = ABC \qquad R \cdot Y_2Z_0 = HCB$$
$$R \cdot Y_1Z_0 = CHA \qquad R \cdot Y_2Z_1 = BAH$$

FIGURE 37

Join O the circumcentre to A B C
Then OA OB OC are respectively perpendicular
to YZ ZX XY
therefore $2AZOY = OA \cdot YZ$

$2BXOZ = OB \cdot ZX$

$2CYOX = OC \cdot XY$

therefore $2\triangle = R\ (YZ + ZX + XY)$

---

* The property that Y_1Z_1 is equal to the semiperimeter of XYZ occurs in Lhuilier's *Élémens d'Analyse*, p. 231 (1809)

† The first of these equalities is given by Feuerbach, *Eigenschaften des ... Dreiecks*, § 19, or Section VI., Theorem 3 (1822). The other three are given by C. Hellwig in Grunert's *Archiv*, XIX., 27 (1852). The proof is that of Messrs W. E. Heal and P. F. Mange in Artemas Martin's *Mathematical Visitor*, II. 42 (1883)

(7) *The following triangles are isosceles* :

$X'ZZ_2 \quad X'Z_2Y \quad X'ZY_3 \quad X'Y_3Y$

For triangles YZZ_2 YZY_3 are right-angled and X′ is the mid point of their hypotenuse

Similarly there are four isosceles triangles with vertex Y′
and „ „ „ „ „ Z′

(8) Y_1Z_1 is antiparallel to BC with respect to A
Z_2X_2 „ „ „ CA „ „ „ B
X_3Y_3 „ „ „ AB „ „ „ C

(9) Y_3Z_2 is parallel to BC
Z_1X_3 „ „ „ CA
X_2Y_1 „ „ „ AB

(10) *Z_1X_3 and Y_1X_2 intersect on the symmedian from A*

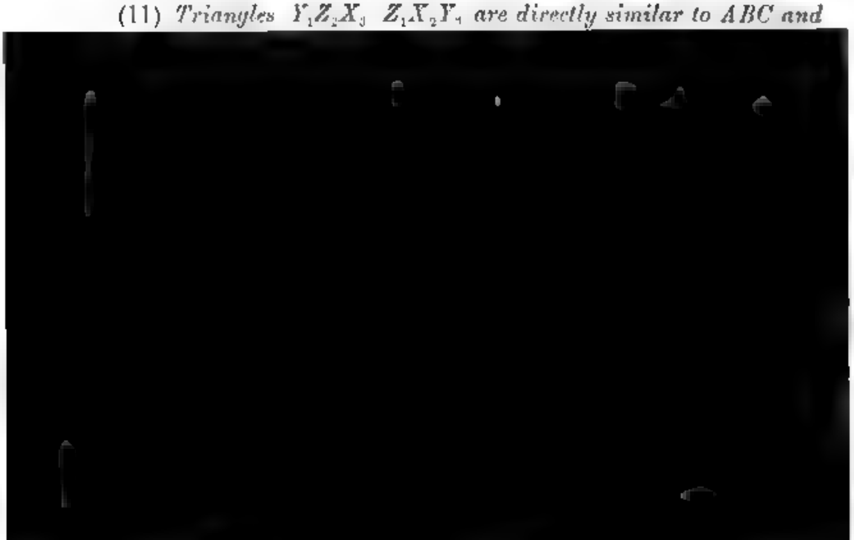
Let A_1 be their point of intersection

Then $AZ_1A_1Y_1$ is a parallelogram, and AA_1 bisects Y_1Z_1

But Y_1Z_1 is antiparallel to BC with respect to A; therefore AA_1 is the symmedian from A

Similarly Y_1X_2 Z_2Y_3 intersect on the symmedian from B;
and Z_2Y_3 Z_1X_3 „ „ „ „ „ C

(11) *Triangles $Y_1Z_2X_3$ $Z_1X_2Y_3$ are directly similar to ABC and*



(12) Since XYZ is the orthic triangle not only of ABC, but also of HCB CHA BAH, if the projections of X Y Z be taken on the sides of the last three triangles, three other circles are obtained

These circles are

$$X_2X_3Y_2Y_0Z_3Z_0 \qquad X_1X_0Y_3Y_1Z_0Z_3 \qquad X_0X_1Y_0Y_2Z_1Z_2$$

If they be denoted by T_1 T_2 T_3 and the circle

$X_3X_2Y_1Y_3Z_2Z_1$ „ „ T, then

$$\left.\begin{matrix} T & T_1 \\ T & T_2 \\ T & T_3 \\ T_2 & T_3 \\ T_3 & T_1 \\ T_1 & T_2 \end{matrix}\right\} \text{ have for radical axis } \left\{\begin{matrix} BC \\ CA \\ AB \\ AX \\ BY \\ CZ \end{matrix}\right.$$

(13) *The centres of the circles* T T_1 T_2 T_3 *are the incentre and the excentres of the triangle* $X'Y'Z'$

For Y_1Z_1 Z_2X_2 X_3Y_3 are equal chords in circle T;
therefore the centre of T is equidistant from them
But these chords form by their intersection the triangle $X'Y'Z'$;
therefore the centre of T must be the incentre of $X'Y'Z'$

Hence T T_1 T_2 T_3 form an orthic tetrastigm

(14) The centres of T T_1 T_2 T_3 are the four points of concurrency of the triads of perpendiculars from X' Y' Z' on the sides of ABC HCB CHA BAH

See *Proceedings of the Edinburgh Mathematical Society*, I. 66 (1894)

(15) *The circle T belongs to the group of Tucker circles* *

FIGURE 36

For triangle $Z_1X_2Y_3$ is similar to ABC;
and it is inscribed in ABC
Hence its circumcircle T belongs to the group of Tucker circles

* Dr Kiehl of Bromberg. See his *Zur Theorie der Transversalen*, pp. 7-8 (1881). See also *Proceedings of the London Mathematical Society*, XV. 281 (1884)

(16) *The circle T cuts orthogonally the three excircles of the orthic triangle XYZ, and each of the circles T_1 T_2 T_3 cuts orthogonally* the incircle and two of the excircles of XYZ*

FIGURE 36

Let p_1 p_2 p_3 denote the perpendiculars from
A B C on YZ ZX XY;
these perpendiculars are the radii of the three excircles of XYZ

Since triangles AYZ ABC are similar,

therefore $p_1^2 : AX^2 = AZ^2 : AC^2$

therefore $p_1^2 : AC \cdot AY_1 = AC \cdot AY_2 : AC^2$

therefore $p_1^2 = AY_1 \cdot AY_2$ the potency of the point A with respect to the circle T

Hence the circle with centre A and radius p_1 cuts the circle T orthogonally

Similarly for the other statements

(17) *The squares of the radii of the circles* †

	T	T_1	T_2	T_3
are	$\frac{1}{2}(\rho^2+\sigma^2)$	$\frac{1}{2}(\rho_1^2+\sigma_1^2)$	$\frac{1}{2}(\rho_2^2+\sigma_2^2)$	$\frac{1}{2}(\rho_3^2+\sigma_3^2)$

FIGURE 36

The triangle X'Y'Z' is similar to XYZ,

[redacted]

(18) *The sum of the squares of the radii of the circles T T_1 T_2 T_3 is equal to the square of the radius of the circumcircle of ABC*

In reference to triangle ABC, the following property may be proved to be true

$$16R^2 = r^2 + r_1^2 + r_2^2 + r_3^2 + a^2 + b^2 + c^2$$

This becomes in reference to triangle X′Y′Z′

$$16(\tfrac{1}{4}R)^2 = (\tfrac{1}{2}\rho)^2 + (\tfrac{1}{2}\rho_1)^2 + (\tfrac{1}{2}\rho_2)^2 + (\tfrac{1}{2}\rho_3)^2 + (\tfrac{1}{2}x)^2 + (\tfrac{1}{2}y)^2 + (\tfrac{1}{2}z)^2$$

$$\text{or} \quad R^2 = \tfrac{1}{4}(\rho^2 + \rho_1^2 + \rho_2^2 + \rho_3^2) + \tfrac{1}{4}(x^2 + y^2 + z^2)$$

$$= \tfrac{1}{4}(\rho^2 + \rho_1^2 + \rho_2^2 + \rho_3^2) + \tfrac{1}{4}(\sigma^2 + \sigma_1^2 + \sigma_2^2 + \sigma_3^2)$$

In connection with the Taylor circles it may be interesting to compare the properties given in the *Proceedings of the Edinburgh Mathematical Society*, Vol. I. pp. 88-96 (1894). These properties were worked out before the Taylor circle had attracted much attention.

(19) *If $A'B'C'$ be the complementary, and XYZ the orthic triangle of ABC, the Wallace lines of the points A' B' C' with respect to the triangle XYZ pass through the centre of the circle T*

FIGURE 36

It is well known that the points A′ B′ C′ X Y Z are situated on the nine point circle of ABC

Since A′ is the mid point of BC

therefore $\qquad A'Y = A'Z$

therefore the foot of the perpendicular from A′ on YZ is X′ the mid point of YZ

Since BC bisects the exterior angle between XY and ZX the straight line joining the feet of the perpendiculars from A′ on XY and ZX will be perpendicular to BC

Hence the Wallace line A′ (XYZ) passes through X′ and is perpendicular to BC

that is, it passes through the centre of T

Similarly for the Wallace lines B′ (XYZ) and C′ (XYZ)

(20) *The Wallace lines of the points X Y Z with respect to the triangle A′B′C′ pass through the centre of the circle T*

[The reader is requested to make the figure]

Let the feet of the perpendiculars from X on B′C′ C′A′ A′B′ be L M N

Then the points A′ M X N are concyclic

therefore $\angle A'MN = \angle A'XN$

$= 90° - \angle B$

$= \angle CAO$

therefore LMN is parallel to AO

But L is the mid point of AX and H_1 is situated on AO therefore LMN passes through the mid point of H_1X, that is through T

(21) *If* H_1 H_2 H_3 *be the orthocentres of triangles* AYZ ZBX XYC *the lines* H_1X H_2Y H_3Z *pass through the centre of circle T and are there bisected*

FIGURE 36

The orthocentre H_1 of triangle AYZ is the point of intersection

[redacted]

Not only is XYZ the orthic triangle of ABC and

	triangles		similar to
AYZ	XBZ	XYC	ABC

but XYZ is the orthic triangle of HCB CHA BAH and

	triangles		similar to
HYZ	XCZ	XYB	HCB
CYZ	XHZ	XYA	CHA
BYZ	XAZ	XYH	BAH

Let the orthocentres of the second, third, and fourth triads of triangles be denoted by

$$H_1' \quad H_2' \quad H_3' \qquad H_1'' \quad H_2'' \quad H_3'' \qquad H_1''' \quad H_2''' \quad H_3'''$$

The following results (among several others) will be found to be established in the *Proceedings of the Edinburgh Mathematical Society*, I. 83–87 (1894). They are quoted here, without proof, to save the reader the trouble or the expense of hunting out the reference

(24) The homothetic centre of the triangles

XYZ	$H_1' H_2' H_3'$	is	T_1
XYZ	$H_1'' H_2'' H_3''$	,,	T_2
XYZ	$H_1''' H_2''' H_3'''$	,,	T_3

and $T_1T_2T_3$ is similar and oppositely situated to ABC

(25) The point T is the centre of three parallelograms

$$YZH_2H_3 \qquad ZXH_3H_1 \qquad XYH_1H_2$$

and similarly T_1 T_2 T_3 are each the centre of three parallelograms

Let the incircle and the excircles of XYZ be denoted by their centres H A B C

(26) The radical axes of

	H A	H B	H C	B C	C A	A B
are	T_2T_3	T_3T_1	T_1T_2	T_1T	T_2T	T_3T

(27) The radical centres of

	A B C	H C B	C H A	B A H
are	T	T_1	T_2	T_3

(28) X′ Y′ Z′ are the feet of the perpendiculars of the triangle $T_1T_2T_3$

(29) The homothetic centre of the triangles

$T_1T_2T_3$	$H_1'H_1''H_1'''$	is	X
$T_1T_2T_3$	$H_2'H_2''H_2'''$	„	Y
$T_1T_2T_3$	$H_3'H_3''H_3'''$	„	Z

(30) The straight lines

HT AT_1 BT_2 CT_3

pass through the centroid of XYZ

(31) If G′ denote this centroid

$$HG' : TG' = AG' : T_1G' = BG' : T_2G' = CG' : T_3G'$$
$$= 2 : 1$$

(32) If HG′T be produced to J′ so that TJ′ = HT then J′ will be the incentre $X_1Y_1Z_1$ the triangle anticomplementary to XYZ

Similarly J_1' J_2' J_3' situated on AT_1 BT_2 CT_3 so that $T_1J_1' = AT_1$ and so on, will be the first, second, and third excircles of $X_1Y_1Z_1$

FIGURE 38

Let X X' Y Y' Z Z' be the six points

Join LL' MM' NN'

The complete quadrilateral $A F \Gamma E B C$ has its diagonal $A\Gamma$ cut harmonically by FE BC;
therefore A U Γ D is a harmonic range;
therefore $E \cdot A$ U Γ D is a harmonic pencil

Now $\Gamma M E M'$ is a parallelogram;
therefore MM' is bisected by $E\Gamma$
therefore MM' is parallel to that ray of the harmonic pencil which is conjugate to $E\Gamma$, namely EA

In like manner NN' is parallel to AB, and LL' to BC

Again, since $YEM'M$ $Y'EMM'$ are parallelograms,

therefore $$YE = Y'E$$

Similarly $$Z'F = ZF$$

therefore $$YE : Y'E = Z'F : ZF$$

Now YZ' is parallel to EF;
therefore $Y'Z$ is parallel to EF

In like manner $Z'X$ is parallel to FD and $X'Y$ to DE

Hence the two hexagons $LL'MM'NN'$ and $XX'YY'ZZ'$ are similar, and the ratio of their corresponding sides is that of 1 to 2

Lastly, since LL' is parallel to BC

$$\begin{aligned} \angle L'L\Gamma &= \angle CDE \\ &= \angle CED \\ &= \angle MM'\Gamma \end{aligned}$$

therefore the points L L' M M' are concyclic
Similarly the points M M' N N' are concyclic
and the points N N' L L' are concyclic;
therefore all the six points are concyclic *

Hence the six points X X' Y Y' Z Z' are also concyclic

---

* This method of proof is different from Adams's

(1) *The centre of Adams's circle is the incentre* of ABC*

For XX′ YY′ ZZ′ are chords of Adams's circle, and they are bisected at D E F;
hence the centre of Adams's circle is found by drawing through D E F perpendiculars to XX′ YY′ ZZ′
These perpendiculars are concurrent at I the incentre of ABC

(2) *To find the centre of the circle LL′MM′NN′*

Since Γ is the homothetic centre of the two circles XX′YY′ZZ′ and LL′MM′NN′, and I is the centre of the first of these circles, therefore the centre of the second circle is situated on ΓI

If I′ denote the centre of the second circle

then $$\Gamma I : \Gamma I' = 2 : 1$$

(3) Since Γ the Gergonne point of ABC is the insymmedian point of DEF, the circle LL′MM′NN′ is the triplicate ratio or first Lemoine circle of DEF

(4) Besides the six-point circle obtained by drawing through Γ the Gergonne point of ABC parallels to the sides of triangle DEF, three other six-point circles will be obtained if through the associated Gergonne points Γ_1 Γ_2 Γ_3 parallels be drawn to the sides of the triangles $D_1E_1F_1$ $D_2E_2F_2$ $D_3E_3F_3$ respectively

The centres of these three circles are the excentres of ABC

[redacted]

$$\left.\begin{array}{lll} BR' = \dfrac{ac^2}{c^2 - b^2} & CS' = \dfrac{ba^2}{c^2 - a^2} & AT' = \dfrac{cb^2}{b^2 - a^2} \\ CR' = \dfrac{ab^2}{c^2 - b^2} & AS' = \dfrac{bc^2}{c^2 - a^2} & BT' = \dfrac{ca^2}{b^2 - a^2} \end{array}\right\} \quad (2)$$

$$RR' = \frac{2ab^2c^2}{c^4 - b^4} \qquad SS' = \frac{2a^2bc^2}{c^4 - a^4} \qquad TT' = \frac{2a^2b^2c}{b^4 - a^4} \qquad (3)$$

Let the three internal medians be denoted by

$$m_1 \qquad m_2 \qquad m_3$$

Their values in terms of the sides are

$$4m_1^2 = -a^2 + 2b^2 + 2c^2$$
$$4m_2^2 = 2a^2 - b^2 + 2c^2$$
$$4m_3^2 = 2a^2 + 2b^2 - c^2$$

$$AR = \frac{2bc\,m_1}{b^2 + c^2} \qquad BS = \frac{2ca\,m_2}{c^2 + a^2} \qquad CT = \frac{2ab\,m_3}{a^2 + b^2} \qquad (4)$$

FIGURE 12

Let AA′ AR be the internal median and symmedian from A

Then $\qquad BR \cdot CR : BA' \cdot CA' = AR^2 : AA'^2$

therefore
$$AR^2 = \frac{BR \cdot CR}{BA' \cdot CA'} \cdot AA'^2$$

$$AR' = \frac{abc}{c^2 - b^2} \qquad BS' = \frac{abc}{c^2 - a^2} \qquad CT' = \frac{abc}{b^2 - a^2} \qquad (5)$$

FIGURE 14

For $\qquad AR'^2 = BR' \,.\, CR'$

$$(AR^2 + BR^2)b^2 + (AR^2 + CR^2)c^2 = 2b^2c^2 \qquad (6)\,*$$

and so on

* Mr Clément Thiry, *Applications remarquables du Théorème de Stewart*, p. 20 (1891)

$$\left.\begin{aligned}
AK &= \frac{2bc\, m_1}{a^2+b^2+c^2} & RK &= \frac{2a^2bc\, m_1}{(b^2+c^2)(a^2+b^2+c^2)} \\
BK &= \frac{2ca\, m_2}{a^2+b^2+c^2} & SK &= \frac{2ab^2c\, m_2}{(c^2+a^2)(a^2+b^2+c^2)} \\
CK &= \frac{2ab\, m_3}{a^2+b^2+c^2} & TK &= \frac{2abc^2\, m_3}{(a^2+b^2)(a^2+b^2+c^2)}
\end{aligned}\right\} \quad (7)^*$$

Figure 18

Since $$\frac{AS}{CS}=\frac{AKB}{CKB} \qquad \frac{AT}{BT}=\frac{AKC}{BKC}$$

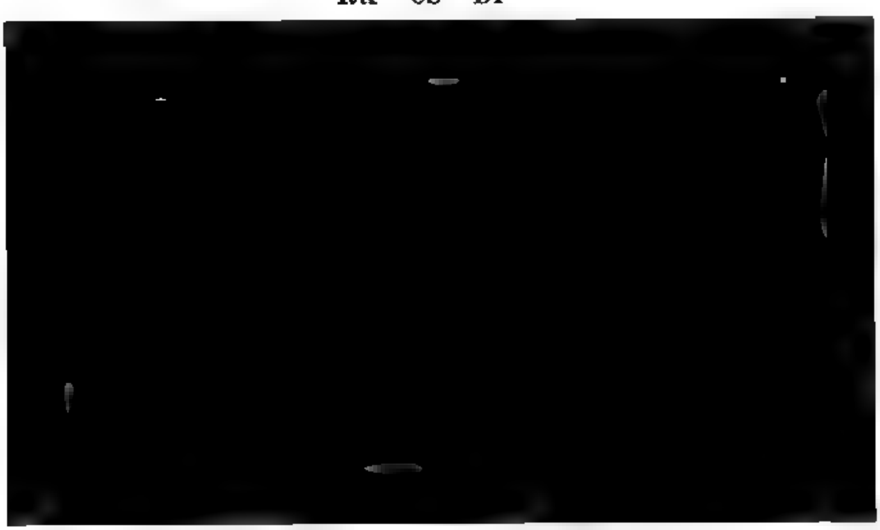
therefore $$\frac{AS}{CS}+\frac{AT}{BT}=\frac{AKB+AKC}{BKC}$$

$$=\frac{AKB+AKC}{BKR+CKR}$$

Now $$\frac{AK}{RK}=\frac{AKB}{BKR}=\frac{AKC}{CKR}$$

$$=\frac{AKB+AKC}{BKR+CKR}$$

therefore $$\frac{AK}{RK}=\frac{AS}{CS}+\frac{AT}{BT}$$



$$\left.\begin{aligned}
AK_1 &= \frac{2bc\,m_1}{-a^2+b^2+c^2} & RK_1 &= \frac{2a^2bc\,m_1}{(b^2+c^2)(-a^2+b^2+c^2)} \\
BK_2 &= \frac{2ca\,m_2}{a^2-b^2+c^2} & SK_2 &= \frac{2ab^2c\,m_2}{(c^2+a^2)(a^2-b^2+c^2)} \\
CK_3 &= \frac{2ab\,m_3}{a^2+b^2-c^2} & TK_3 &= \frac{2abc^2\,m_3}{(a^2+b^2)(a^2+b^2-c^2)}
\end{aligned}\right\} \quad (10)$$

$$\left.\begin{aligned}
BK_1 = CK_1 &= \frac{abc}{-a^2+b^2+c^2} \\
CK_2 = AK_2 &= \frac{abc}{a^2-b^2+c^2} \\
AK_3 = BK_3 &= \frac{abc}{a^2+b^2-c^2}
\end{aligned}\right\} \quad (11)$$

$$\left.\begin{aligned}
KK_1 &= \frac{4a^2bc\,m_1}{(a^2+b^2+c^2)(-a^2+b^2+c^2)} \\
KK_2 &= \frac{4ab^2c\,m_2}{(a^2+b^2+c^2)(a^2-b^2+c^2)} \\
KK_3 &= \frac{4abc^2\,m_3}{(a^2+b^2+c^2)(a^2+b^2-c^2)}
\end{aligned}\right\} \quad (12)$$

$$\left.\begin{aligned}
K_2K_3 &= \frac{2a^3bc}{(a^2-b^2+c^2)(a^2+b^2-c^2)} \\
K_3K_1 &= \frac{2ab^3c}{(a^2+b^2-c^2)(-a^2+b^2+c^2)} \\
K_1K_2 &= \frac{2abc^3}{(-a^2+b^2+c^2)(a^2-b^2+c^2)}
\end{aligned}\right\} \quad (13)$$

$$a^2\frac{AK_1}{KK_1} = b^2\frac{BK_2}{KK_2} = c^2\frac{CK_3}{KK_3} = \frac{a^2+b^2+c^2}{2} \quad (14)^*$$

$$\frac{KK_1}{AK_1} : \frac{KK_2}{BK_2} : \frac{KK_3}{CK_3} = a^2 : b^2 : c^2 \quad (15)^*$$

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* Dr Franz Wetzig in Schlömilch's *Zeitschrift*, XII. 294 (1867)

$$\left.\begin{aligned}
BL &= \frac{a(a^2 - b^2 + 3c^2)}{2(a^2+b^2+c^2)} = \frac{a(c^2 + ca\cos B)}{a^2+b^2+c^2} \\
CL &= \frac{a(a^2 + 3b^2 - c^2)}{2(a^2+b^2+c^2)} = \frac{a(b^2 + ab\cos C)}{a^2+b^2+c^2} \\
CM &= \frac{b(3a^2 + b^2 - c^2)}{2(a^2+b^2+c^2)} = \frac{b(a^2 + ab\cos C)}{a^2+b^2+c^2} \\
AM &= \frac{b(-a^2 + b^2 + 3c^2)}{2(a^2+b^2+c^2)} = \frac{b(c^2 + bc\cos A)}{a^2+b^2+c^2} \\
AN &= \frac{c(-a^2 + 3b^2 + c^2)}{2(a^2+b^2+c^2)} = \frac{c(b^2 + bc\cos A)}{a^2+b^2+c^2} \\
BN &= \frac{c(3a^2 - b^2 + c^2)}{2(a^2+b^2+c^2)} = \frac{c(a^2 + ca\cos B)}{a^2+b^2+c^2}
\end{aligned}\right\} \quad (16)\,*$$

Distances of K from the sides of ABC

$$\left.\begin{aligned}
KL &= \frac{2a\Delta}{a^2+b^2+c^2} = \frac{a\sin A\sin B\sin C}{\sin^2 A + \sin^2 B + \sin^2 C} \\
KM &= \frac{2b\Delta}{a^2+b^2+c^2} = \frac{b\sin A\sin B\sin C}{\sin^2 A + \sin^2 B + \sin^2 C}
\end{aligned}\right\} \quad (17)\,*$$

KN = $2c\Delta$ … = $c\sin A \sin B \sin C$ …

[redacted]

The following is another demonstration *

Let α β γ denote the distances of K from BC CA AB

Then
$$\frac{\alpha}{a}=\frac{\beta}{b}=\frac{\gamma}{c}$$
$$=\frac{a\alpha}{a^2}=\frac{b\beta}{b^2}=\frac{c\gamma}{c^2}$$
$$=\frac{a\alpha+b\beta+c\gamma}{a^2+b^2+c^2}$$
$$=\frac{2\Delta}{a^2+b^2+c^2}$$

$$\left.\begin{aligned}\mathrm{KL}^2+\mathrm{KM}^2+\mathrm{KN}^2 &= \frac{4\Delta^2}{a^2+b^2+c^2}\\ =\frac{a^2\sin^2\mathrm{B}\sin^2\mathrm{C}}{\sin^2\mathrm{A}+\sin^2\mathrm{B}+\sin^2\mathrm{C}} &= \frac{b^2\sin^2\mathrm{C}\sin^2\mathrm{A}}{\sin^2\mathrm{A}+\sin^2\mathrm{B}+\sin^2\mathrm{C}}\\ =\frac{c^2\sin^2\mathrm{A}\sin^2\mathrm{B}}{\sin^2\mathrm{A}+\sin^2\mathrm{B}+\sin^2\mathrm{C}} &= \frac{\Delta}{\cot\mathrm{A}+\cot\mathrm{B}+\cot\mathrm{C}}\end{aligned}\right\}(18)\dagger$$

Distances of K_1 K_2 K_3 from the sides of ABC

$$\left.\begin{array}{cccc}(\mathrm{K}_1) & \dfrac{-2a\Delta}{-a^2+b^2+c^2} & \dfrac{2b\Delta}{-a^2+b^2+c^2} & \dfrac{2c\Delta}{-a^2+b^2+c^2}\\[2ex] (\mathrm{K}_2) & \dfrac{2a\Delta}{a^2-b^2+c^2} & \dfrac{-2b\Delta}{a^2-b^2+c^2} & \dfrac{2c\Delta}{a^2-b^2+c^2}\\[2ex] (\mathrm{K}_3) & \dfrac{2a\Delta}{a^2+b^2-c^2} & \dfrac{2b\Delta}{a^2+b^2-c^2} & \dfrac{-2c\Delta}{a^2+b^2-c^2}\end{array}\right\}\quad(19)$$

* Mr R. Tucker in *Quarterly Journal of Mathematics*, XIX. 342 (1883)

† The first of these values is given by "Yanto" in Leybourn's *Mathematical Repository*, old series, Vol. III. p. 71 (1803). Lhuilier in his *Élémens d'Analyse*, p. 298 (1809) gives the analogous property for the tetrahedron.

The other values are given by E. W. Grebe in Grunert's *Archiv*, IX. 251 (1847)

Grebe, *loco citato*, p. 257, gives the distances of K_3 from the sides of ABC with the following trigonometrical equivalents

$$\left.\begin{aligned}
\frac{a \sin A \sin B \sin C}{\sin^2 A + \sin^2 B - \sin^2 C} &= \tfrac{1}{2} a \tan C \\
\frac{b \sin A \sin B \sin C}{\sin^2 A + \sin^2 B - \sin^2 C} &= \tfrac{1}{2} b \tan C \\
\frac{-c \sin A \sin B \sin C}{\sin^2 A + \sin^2 B - \sin^2 C} &= -\tfrac{1}{2} c \tan C
\end{aligned}\right\} \quad (20)$$

If Σ Σ_1 Σ_2 Σ_3 denote the sum of the squares of the distances from the sides of ABC of K K_1 K_2 K_3

$$\Sigma_1 = \frac{4\Delta^2}{-a^2+b^2+c^2} \qquad \Sigma_2 = \frac{4\Delta^2}{a^2-b^2+c^2} \qquad \Sigma_3 = \frac{4\Delta^2}{a^2+b^2-c^2} \quad (21)\,*$$

$$\frac{1}{\Sigma} = \frac{1}{\Sigma_1} + \frac{1}{\Sigma_2} + \frac{1}{\Sigma_3} \quad (22)\,*$$

If k k_1 k_2 k_3 denote the distances from BC of K K_1 K_2 K_3

$$\frac{1}{k} + \frac{1}{k_1} = \frac{1}{k_2} + \frac{1}{k_3} = \frac{2}{h_1} \quad (23)\,*$$

This relation holds for any four harmonically associated points

[redacted]

For MN can be found by applying Ptolemy's theorem to the encyclic quadrilateral ANKM

$$\text{LMN} = \frac{12\Delta^3}{(a^2+b^2+c^2)^2} \qquad (26)*$$

FIGURE 28

Since K is the controid of LMN,

$$\text{LMN} = 3\text{KLK}'$$

Now KLK′ has its sides equal to KL KM KN and it is similar to ABC

therefore $$\frac{\text{KLK}'}{\text{ABC}} = \frac{\text{KL}^2+\text{KM}^2+\text{KN}^2}{a^2+b^2+c^2}$$

$$= \frac{4\Delta^2}{(a^2+b^2+c^2)^2}$$

$$\text{KBC} : \text{KCA} : \text{KAB} = a^2 : b^2 : c^2 \qquad (27)\dagger$$

$$\left.\begin{aligned} \text{K}_1\text{BC} : \text{K}_1\text{CA} : \text{K}_1\text{AB} &= -a^2 : b^2 : c^2 \\ \text{K}_2\text{BC} : \text{K}_2\text{CA} : \text{K}_2\text{AB} &= a^2 : -b^2 : c^2 \\ \text{K}_3\text{BC} : \text{K}_3\text{CA} : \text{K}_3\text{AB} &= a^2 : b^2 : -c^2 \end{aligned}\right\} (28)\ddagger$$

$$\left.\begin{aligned} \text{AA}'\cdot\text{BB}'\cdot\text{CC}' &: \text{AK}_1\cdot\text{BK}_2\cdot\text{CK}_3 \\ &= \text{AK}\cdot\text{BK}\cdot\text{CK} : \text{KK}_1\cdot\text{KK}_2\cdot\text{KK}_3 \end{aligned}\right\} (29)\ddagger$$

$$\left.\begin{aligned} \text{AK}_2 : \text{AK}_3 &= \text{CX} : \text{BX} \\ \text{BK}_3 : \text{BK}_1 &= \text{AY} : \text{CY} \\ \text{CK}_1 : \text{CK}_2 &= \text{BZ} : \text{AZ} \end{aligned}\right\} \qquad (30)\ddagger$$

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* Dr Franz Wetzig in Schlömilch's *Zeitschrift*, XII. 298 (1867)

† L. C. Schulz von Strasznicki in Baumgartner and D'Ettingshausen's *Zeitschrift für Physik und Mathematik*, II. 403 (1827)

‡ Dr Franz Wetzig in Schlömilch's *Zeitschrift*, XII. 287, 293, 291 (1867)

FIGURE 28

Draw CZ perpendicular to AB

Let the tangent at A meet K_1C produced at K_2 and draw K_2B' perpendicular to CA

From the similar triangles K_1CA' CAZ

$$K_1C : CA' = CA : AZ$$

or $$K_1C : a = b : 2AZ$$

From the similar triangles K_2CB' CBZ

$$K_2C : CB' = CB : BZ$$

or $$K_2C : b = a : 2BZ$$

Hence $$K_1C \cdot AZ = \tfrac{1}{2}ab$$
$$= K_2C \cdot BZ$$

therefore $$K_1C : K_2C = BZ : AZ$$

If k_1 k_2 k_3 denote the distances of K from BC CA AB

$$k_1^2 + k_2^2 + k_3^2 : a^2 + b^2 + c^2 = \tfrac{1}{3}LMN : ABC \qquad (31)*$$

If k_1' k_2' k_3' denote the distances of K_1

k_1'' k_2'' k_3'' " " " " K_2

k_1''' k_2''' k_3''' " " " " K_3

from BC CA AB

[redacted]

$$abc : R^2 = 2\Delta K_1K_2K_3 : \text{radius of circle } K_1K_2K_3 \qquad (34)$$

This follows from the preceding since

$$\frac{abc}{2R^2} = \frac{2\Delta}{R}$$

Cosine Circle or Second Lemoine Circle

Figure 19

$$\left.\begin{aligned}
AE &= \frac{2bc^2}{a^2+b^2+c^2} & AF' &= \frac{2b^2c}{a^2+b^2+c^2} \\
BF &= \frac{2ca^2}{a^2+b^2+c^2} & BD' &= \frac{2c^2a}{a^2+b^2+c^2} \\
CD &= \frac{2ab^2}{a^2+b^2+c^2} & CE' &= \frac{2a^2b}{a^2+b^2+c^2}
\end{aligned}\right\} \qquad (35)$$

For triangles AEF′ ABC are similar and AK is a median of AEF′

therefore $\qquad AE : AB = AK : m_1$

$$\left.\begin{aligned}
AE' &= \frac{b(-a^2+b^2+c^2)}{a^2+b^2+c^2} & AF &= \frac{c(-a^2+b^2+c^2)}{a^2+b^2+c^2} \\
BF' &= \frac{c(a^2-b^2+c^2)}{a^2+b^2+c^2} & BD &= \frac{a(a^2-b^2+c^2)}{a^2+b^2+c^2} \\
CD' &= \frac{a(a^2+b^2-c^2)}{a^2+b^2+c^2} & CE &= \frac{b(a^2+b^2-c^2)}{a^2+b^2+c^2}
\end{aligned}\right\} \qquad (36)$$

$$DD' = \frac{a(-a^2+b^2+c^2)}{a^2+b^2+c^2} \quad EE' = \frac{b(a^2-b^2+c^2)}{a^2+b^2+c^2} \quad FF' = \frac{c(a^2+b^2-c^2)}{a^2+b^2+c^2} \qquad (37)$$

$$FD = \frac{4a\Delta}{a^2+b^2+c^2} \quad DE = \frac{4b\Delta}{a^2+b^2+c^2} \quad EF = \frac{4c\Delta}{a^2+b^2+c_2} \qquad (38)$$

For $\qquad FD^2 = E'D^2 - E'F'^2 = E'D^2 - D'D^2$

$$\mathrm{BD}' \cdot \mathrm{CD} = \mathrm{CE}' \cdot \mathrm{AE} = \mathrm{AF}' \cdot \mathrm{BF} \tag{39}$$

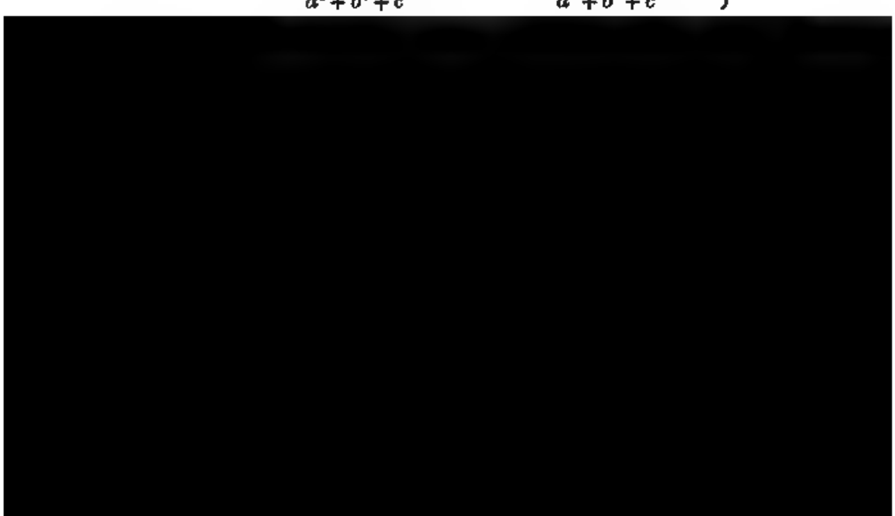
$$\left.\begin{array}{l} \mathrm{BD}' : \mathrm{CE}' : \mathrm{AF}' = \dfrac{c}{b} : \dfrac{a}{c} : \dfrac{b}{a} \\[2ex] \mathrm{BF} : \mathrm{CD} : \mathrm{AE} = \dfrac{a}{b} : \dfrac{b}{c} : \dfrac{c}{a} \end{array}\right\} \tag{40}$$

Triplicate Ratio or First Lemoine Circle

The whole of the subsequent results are taken from two of Mr R. Tucker's papers in the *Quarterly Journal of Mathematics*, XIX. 342-348 (1883) and XX. 57-59 (1885). The proofs are sometimes different from Mr Tucker's

Figure 32

$$\left.\begin{array}{ll} \mathrm{AF} = \dfrac{b^2c}{a^2+b^2+c^2} & \mathrm{AE}' = \dfrac{bc^2}{a^2+b^2+c^2} \\[2ex] \mathrm{BD} = \dfrac{c^2a}{a^2+b^2+c^2} & \mathrm{BF}' = \dfrac{ca^2}{a^2+b^2+c^2} \\[2ex] \mathrm{CE} = \dfrac{a^2b}{a^2+b^2+c^2} & \mathrm{CD}' = \dfrac{ab^2}{a^2+b^2+c^2} \end{array}\right\} \tag{41}$$



$$\left.\begin{array}{ll} AF' = \dfrac{c(b^2+c^2)}{a^2+b^2+c^2} & AE = \dfrac{b(b^2+c^2)}{a^2+b^2+c^2} \\ BD' = \dfrac{a(c^2+a^2)}{a^2+b^2+c^2} & BF = \dfrac{c(c^2+a^2)}{a^2+b^2+c^2} \\ CE' = \dfrac{b(a^2+b^2)}{a^2+b^2+c^2} & CD = \dfrac{a(a^2+b^2)}{a^2+b^2+c^2} \end{array}\right\} \quad (43)$$

$$DD' = \frac{a^3}{a^2+b^2+c^2} \qquad EE' = \frac{b^3}{a^2+b^2+c^2} \qquad FF' = \frac{c^3}{a^2+b^2+c^2} \quad (44)^*$$

$$\left.\begin{array}{l} BD : DD' : D'C = c^2 : a^2 : b^2 \\ CE : EE' : E'A = a^2 : b^2 : c^2 \\ AF : FF' : F'B = b^2 : c^2 : a^2 \end{array}\right\} \quad (45)$$

$$EF' = \frac{a(b^2+c^2)}{a^2+b^2+c^2} \qquad FD' = \frac{b(c^2+a^2)}{a^2+b^2+c^2} \qquad DE' = \frac{c(a^2+b^2)}{a^2+b^2+c^2} \quad (46)$$

$$E'F = F'D = D'E = \frac{abc}{a^2+b^2+c^2} \quad (47)$$

For DE'FF' is a symmetrical trapezium

therefore $E'F^2 = \frac{1}{4}(DE' - FF')^2 + KN^2$

$$= \frac{1}{4}\left\{\frac{c(a^2+b^2)}{a^2+b^2+c^2} - \frac{c^3}{a^2+b^2+c^2}\right\}^2 + \left\{\frac{2c\Delta}{a^2+b^2+c^2}\right\}^2$$

$$= \frac{c^2}{4(a^2+b^2+c^2)^2}\{(a^2+b^2-c^2)^2 + 16\Delta^2\}$$

$$= \frac{4a^2b^2c^2}{4(a^2+b^2+c^2)^2}$$

$$\left.\begin{array}{l} DF = \dfrac{c}{a^2+b^2+c^2}\sqrt{b^2c^2+c^2a^2+a^2b^2} \\ FE = \dfrac{b}{a^2+b^2+c^2}\sqrt{b^2c^2+c^2a^2+a^2b^2} \\ ED = \dfrac{a}{a^2+b^2+c^2}\sqrt{b^2c^2+c^2a^2+a^2b^2} \end{array}\right\} \quad (48)$$

* It was this property which suggested to Mr Tucker the name "triplicate-ratio circle."

For D E′ F F′ are concyclic
and DE′FF′ is a symmetrical trapezium ;

therefore $\qquad DF \cdot E'F' = DE' \cdot FF' + DF' \cdot FE'$

that is $\qquad DF'^2 = DE' \cdot FF' + DF'^2$

$$\left.\begin{aligned} KE \cdot KF' &= KF \cdot KD' = KD \cdot KE' \\ &= E'F'^2 = F'D'^2 = D'E'^2 \end{aligned}\right\} \qquad (49)$$

For $\qquad KE \cdot KF' = CD' \cdot BD$

$$\left.\begin{aligned} &\text{The minimum chord through K} \\ &= 2E'F = 2F'D = 2D'E \end{aligned}\right\} \qquad (50)$$

For $\qquad KE \cdot KF' = E'F'^2$

$$\left.\begin{aligned} DD' \cdot EE' \cdot FF' &= E'F \cdot F'D \cdot D'E \\ &= \frac{a^3b^3c^3}{(a^2+b^2+c^2)^3} \end{aligned}\right\} \qquad (51)$$

The hexagon DD′EE′FF′ has its

perimeter $\qquad = \dfrac{a^3+b^3+c^3+3abc}{a^2+b^2+c^2} \qquad (52)$

[redacted]

$$\text{then} \qquad \left.\begin{array}{lllllll} \mathrm{FL} & : & \mathrm{DL} & = & c^2 & : & a^2 \\ \mathrm{D'L'} & : & \mathrm{E'L'} & = & a^2 & : & b^2 \\ \mathrm{DM} & : & \mathrm{EM} & = & a^2 & : & b^2 \\ \mathrm{E'M'} & : & \mathrm{F'M'} & = & b^2 & : & c^2 \\ \mathrm{EN} & : & \mathrm{FN} & = & b^2 & : & c^2 \\ \mathrm{F'N'} & : & \mathrm{D'N'} & = & c^2 & : & a^2 \end{array}\right\} \qquad (56)$$

FIGURE 33

For $\quad \mathrm{FL} : \mathrm{DL} = \mathrm{FK} : \mathrm{D'K} = \mathrm{AE'} : \mathrm{CE}$

and $\quad \mathrm{D'L'} : \mathrm{E'L'} = \mathrm{CE} : \mathrm{EE'}$

A Certain Linear Differential Equation.

By F. H. Jackson, M.A.

The Series

$$y = 1 + \frac{\Pi(\alpha)\Pi(\beta-m)}{\Pi(\alpha-n)\Pi(\beta)}\cdot\frac{x}{1!} + \frac{\Pi(\alpha)\Pi(\alpha+1)\Pi(\beta-m)\Pi(\beta-m+1)}{\Pi(\alpha-n)\Pi(\alpha-n+1)\Pi(\beta)\Pi(\beta+1)}\cdot\frac{x^2}{2!} + \dots \qquad (1)$$

if convergent, is a particular solution of the Differential Equation

$$\left[(\alpha)_n + n(\alpha)_{n-1}.xD + \frac{n.\overline{n-1}}{2!}(\alpha)_{n-2}x^2D^2 + \quad \dots \quad \dots \right]y$$

$$-\frac{1}{x}\left[(\beta)_m.xD + m(\beta)_{m-1}x^2D^2 + \frac{m.\overline{m-1}}{2!}(\beta)_{m-2}x^3D^3 + \quad \dots \right]y = 0 \qquad (2)$$

in which

$(\alpha)_n \equiv \dfrac{\Pi(\alpha)}{\Pi(\alpha-n)}$ and Π denotes Gauss's Π Function, D stands for $\dfrac{d}{dx}$.

The Differential Equation will contain a finite number or an infinite number of terms according as m and n are, or are not [redacted]

Substituting the values of the differential coefficients in the expression on the left side of equation (2) we have

$$
\begin{aligned}
&(a)_n\Big[A_1x^{m_1} + A_2x^{m_2} + \dots \dots + A_rx^{m_r} + \dots\Big]\\
&+ n.(a)_{n-1}\Big[A_1(m_1)_1x^{m_1} + A_2(m_2)_1x^{m_2} + \dots + A_r(m_r)_1x^{m_r} + \dots\Big]\\
&+ \frac{n.n-1}{2!}(a)_{n-2}\Big[A_1(m_1)_2x^{m_1} + A_2(m_2)_2x^{m_2} + \dots + A_2(m_r)_2x^{m_r} + \dots\Big]\\
&\quad \dots \qquad \dots \qquad \dots\\
&\quad \dots \qquad \dots \qquad \dots\\
&\quad \dots \qquad \dots \qquad \dots\\
&\quad \dots \qquad \dots \qquad \dots\\
&+ \frac{n!}{r!n-r!}(a)_{n-r}\Big[A_1(m_1)_rx^{m_1} + A_2(m_2)_rx^{m_2} + \dots + A_r(m_r)_rx^{m_r} + \dots\Big]\\
&\quad \dots \qquad \dots \qquad \dots\\
&\quad \dots \qquad \dots \qquad \dots\\
&- (\beta)_m\Big[A_1(m_1)_1x^{m_1-1} + A_2(m_2)_1x^{m_2-1} + \dots + A_r(m_r)_1x^{m_r-1} + \dots\Big]\\
&- m(\beta)_{m-1}\Big[A_1(m_1)_2x^{m_1-1} + A_2(m_2)_2x^{m_2-1} + \dots + A_r(m_r)_2x^{m_r-1} + \dots\Big]\\
&\quad \dots \qquad \dots \qquad \dots\\
&\quad \dots \qquad \dots \qquad \dots\\
&\quad \dots \qquad \dots \qquad \dots\\
&\quad \dots \qquad \dots \qquad \dots\\
&- \frac{m!}{r!m-r!}(\beta)_{m-r}\Big[A_1(m_1)_rx^{m_1-1} + A_2(m_2)_rx^{m_2-1} + \dots\\
&\qquad\qquad \dots + A_r(m_r)_rx^{m_r-1} + \dots\Big]\\
&\quad \dots \qquad \dots \qquad \dots\\
&\quad \dots \qquad \dots \qquad \dots
\end{aligned}
$$

This expression must vanish identically and we see that a possible relation between the indices is

$$
\begin{aligned}
m_1 &= m_2 - 1\\
m_2 &= m_3 - 1\\
&\dots\dots\\
&\dots\dots\\
m_r &= m_{r+1} - 1
\end{aligned}
$$

Therefore $m_{r+1} = m_r + 1$ and $m_{r+1} = m_1 + r$.

The coefficients of all the powers of x must vanish separately, the coefficient of x^{m_1-1} is

$$-\mathrm{A}_1\left[(m_1)_1(\beta)_m+m(m_1)_2(\beta)_{m-1}+\ \dots\ +\frac{m!}{r!\,\overline{m-r}!}(m_1)_{r+1}(\beta)_{m-r}+\quad\dots\right]$$

which may be written

$$-\mathrm{A}_1m_1\left[(\beta)_m+m(\beta)_{m-1}(m_1-1)_1+\dots+\frac{m!}{r!\,\overline{m-r}!}(\beta)_{m-r}(m_1-1)_r+\quad\dots\right] \tag{3}$$

This series consists of a finite number of terms, only when m is a positive integer; it is convergent however for all values of m provided that $\beta+m_1>0$; and Expression (3) reduces to $-\mathrm{A}_1m_1(\beta+m_1-1)_m$ (*Proc. Lon. Math. Soc.*, Vol. XXVI. p. 285). Now A_1 is not zero and we see that a possible value of m_1 is zero, other values of m_1 are the roots in m_1 of

$$(\beta+m_1-1)_m=0$$

that is of $$\underset{\kappa=\infty}{\mathrm{L}}\ \frac{(\beta+m_1-m)(\beta+m_1-m+1)\ \dots\ (\beta+m_1-m+\kappa)}{(\beta+m_1)(\beta+m_1+1)\quad\dots\quad(\beta+m_1+\kappa)}.\kappa^m=0$$

$\therefore$ the other values of m_1 are

$$m-\beta,\ m-\beta-1,\ m-\beta-2,\ \text{etc.},\ \dots\ m-\beta-\kappa.$$

The coefficient of x^{m_r} is

[redacted]

Expression (6) which shows the relation between successive coefficients of series (A) is only valid subject to the conditions

$$\left.\begin{aligned} a+m_r+1 &> 0 \\ \beta+m_r+1 &> 0 \end{aligned}\right\} \quad \cdots \quad \cdots \quad (7)$$

Now $m_r = m_1 + r - 1$ and the possible values of m_1 are zero and $m - \beta - s$, where s is zero or any positive integer.

When $m_1 = 0$

The conditions (7) become $\begin{aligned} a+r &> 0 \\ \beta+r &> 0 \end{aligned}$

and since the least value of r is 1

$$\begin{aligned} a+1 &> 0 \\ \beta+1 &> 0 \end{aligned}$$

Subject to these conditions

$$y = A_1\Big[1 + \frac{(a)_n}{(\beta)_m}\cdot\frac{x}{1!} + \frac{(a)_n(a+1)_n}{(\beta)_m(\beta+1)_m}\cdot\frac{x^2}{2!} + \quad \cdots \quad \cdots$$

$$\cdots \quad + \frac{(a)_n(a+1)_n \ \cdots \ (a+r)_n}{(\beta)_m(\beta+1)_m \ \cdots \ (\beta+r)_m}\cdot\frac{x^{r+1}}{r+1!} + \quad \cdots \quad \Big] \quad (8)$$

is a solution of the Differential Equation (2) provided the series on the Right side of (8) is convergent.

When $m_1 = m - \beta - s$, the conditions become

$$\begin{aligned} a+m-\beta-s+r &> 0 \\ \beta+m-\beta-s+r &> 0 \end{aligned}$$

Now the least value of r is 1, therefore

$$\left.\begin{aligned} s &< a-\beta+m+1 \\ s &< \qquad m+1 \end{aligned}\right\} \quad \cdots \quad \cdots \quad (9)$$

and subject to these conditions

$$y = A_s x^{m-\beta-s}\Big[1 + \frac{(a+m-\beta-s)_n}{(m-s)_m}\cdot\frac{x}{(m-\beta-s+1)} + \quad \cdots$$

$$+ \frac{(a+m-\beta-s)_n \ldots (a+m-\beta-s+r)_n}{(m-s)_m \ \ldots \ (m-s+r)_m}\cdot\frac{x^{r+1}}{(m-\beta-s+1)\ldots(m-\beta-s+r+1)} + \cdots$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \Big] \quad \cdots \quad (10)$$

s being zero or any positive integer subject to the conditions (9) is a solution of the Differential Equation (2).

If $\alpha = n$ and $\beta = m$.

The Series (8) becomes

$$y = A_1\left[1 + \frac{(n)_n}{(m)_m}\cdot\frac{x}{1!} + \frac{(n)_n(n+1)_n}{(m)_m(m+1)_m}\cdot\frac{x^2}{2!} + \quad \dots \quad \dots \quad \dots \right]$$

$$= A_1\left[1 + \frac{\Pi(n)}{\Pi(m)}\cdot\frac{x}{1!} + \frac{\Pi(n).\Pi(n+1)}{\Pi(m)\Pi(m+1)}\cdot\frac{x^2}{2!} + \dots \quad \dots \quad \dots \right] \quad (11)$$

The Series (10) becomes

$$y = A_s x^{-s}\left[1 + \frac{(n-s)_n}{(m-s)_m}\cdot\frac{x}{1-s} + \frac{(n-s)_n(n-s+1)_n}{(m-s)_m(m-s+1)_m}\cdot\frac{x^2}{1-s.2-s} + \dots\right] \quad (12)$$

s being zero or any positive integer subject to the conditions

$$s < m + 1$$

$$s < n + 1$$

These series are solutions of the Differential Equations

$$\Pi(n)\left[1 + \frac{n}{(1!)^2}xD + \frac{n.n-1}{(2!)^2}x^2D^2 + \quad \dots \quad \dots \right]y$$

$$-\frac{1}{x}\Pi(m)\left[xD + \frac{m}{(1!)^2}x^2D^2 + \frac{m.m-1}{(2!)^2}x^3D^3 + \quad \dots \quad \dots \right]y = 0 \quad (13)$$

a particular case of (2) when $\alpha = n$, $\beta = m$.

On Superposition by the Aid of Dissection.

By R. F. Muirhead, M.A., B.Sc.

What do we mean when we assert that one plane figure is equal to another? In trying to find a satisfactory answer to this question I was led to consider the subjects treated in this paper.

Euclid's Axiom : *Magnitudes which coincide are equal*, may be taken as defining the nature of geometrical equality: but it obviously does not apply to any but congruent figures. We must therefore have a more comprehensive definition. I suppose the conception tacitly used by mathematicians is that magnitudes are equal which can be so dissected that for each part of one there is a corresponding part of the other which is congruent to it. This would be sufficient for elementary Geometry, while for higher Geometry the method of limits would be needed in addition, and magnitudes would have to be recognised as equal if pairs of mutually congruent parts could be subtracted from them continually until the remainders were both infinitesimal.

Now the first proposition of Euclid in which equal non-congruent figures occur, is the 35th of Book I. But there it is to be noted that the two parallelograms are proved to be equal, not directly by dissecting them into mutually congruent parts, but by the aid of the axiom that *if equals be taken from equals, the remainders are equal.* However, a very simple method of dissection rendering superposition possible in this case is well known, and is given in some school editions of Euclid.

The more general question arises, as to whether any two equal rectilinear plane figures can be rendered superposable by dissection into a finite number of parts. I shall show that this question may be answered in the affirmative.

Prop. 1. Any two rectilinear figures, or systems of rectilinear figures, A and B, can be dissected into pairs of equal triangles, one triangle of each pair belonging to A and the other to B.

By joining vertices, we can divide both A and B into triangular areas. Suppose the system A gives m, and the system B, n triangles. Consider any pair of triangles, one from A and the other from B. If they are equal, cut off both, and A is left with $m-1$, B with $n-1$ triangles. But if the triangle in A is greater than that in B,

cut off from the former a part equal to the latter, and cut off the latter, and A is left with m, B with $n-1$ triangles. Thus the total number of triangles is diminished by *two* or *one* according as the triangles chosen are equal or unequal. By repeating this process, we dissect A and B as required, and the number of pairs of congruent triangles cannot exceed $m+n$.

Prop. 2. Any two equal triangles can be made superposable by dissection.

Case 1. If the triangles have two sides of one equal to two sides of the other, and the contained angles supplementary, they may be placed as the triangles ABC, DBC in Fig. 40. Then by joining the mid points of AB and BD to C we divide the triangles into two pairs of congruent triangles marked 1, 1 and 2, 2.

Case 2. If the triangles have a side of one equal to a side of the other, let them be placed with this as common base, but on opposite sides of it as in Fig. 41, where ABC, DBC are the equal triangles.

Join AD which will be bisected by BC in E. Then ABE, DBE are triangles related as in Case 1, and can therefore be dissected as required. So also the triangles AEC, DEC. Thus we have four pairs of congruent triangles 1, 1; 2, 2; 3, 3; and 4, 4.

But this simple construction fails when AD falls without the given triangles. In such a case, as in Fig. 42, where the line AD cuts BC produced in E, a further construction is necessary.

As before, join E to the mid points of AB and AC. Then take [redacted]

part of P be then dissected in the same manner as the corresponding part of R is dissected by the lines of the second part of the construction, and let the parts of Q be similarly treated. Then P and Q are obviously dissected as required.

Prop. 3. (Corollary to the foregoing). Any two plane rectilinear figures of equal area can be made superposable by dissection.

I made some attempts to carry out similar dissections for solid figures bounded by plane faces, but though I succeeded in the case of parallelepipeds and prisms, I failed in the case of tetrahedra and other pyramids. It is to be remarked that precisely at this point in the theory of solids does Euclid discontinue the use of the elementary methods of Book XI., relegating the theory of pyramids (with that of circular areas, etc.) to Book XII. where a method of Exhaustions is used.

It seems to me probable for certain reasons that where I failed, the problem is insoluble, *i.e.*, that in general it is impossible to render two tetrahedra of equal volume congruent by means of dissection into a finite number of parts, and I am not sure but that this impossibility may be conclusively demonstrable. So far, however, I have not arrived at a satisfactory demonstration.

Probably, also, there is a connection between the "superposability" of two solids of equal volume or of two plane figures of equal area and the possibility of proving their equality by elementary geometrical methods, such that one involves the other.

By "superposability" I here mean capability of dissection into a finite number of pairs of congruent parts, one from each figure.

To prove this connection, it would be necessary to show that all the axioms relating to geometrical equality hold good also with regard to "superposability," using this word in the sense just explained. For example we should have to show that *if "superposables" be added to "superposables," the wholes are "superposable,"* etc. I find that the only axiom which presents any difficulty is that corresponding to Euclid's Axiom 3, viz: *if "superposables" be taken from "superposables" the remainders are "superposable,"* which again can be made to depend on this: *if congruents be taken from congruents the remainders are "superposable."* I have not yet succeeded in establishing this generally, but I find that a construction, of which Fig. 42 is a particular case, goes a good way towards

it: in fact it demonstrates the axiom for *solids* with the exception of a special class of cases. But into these questions I shall not enter further at present.

After I had obtained practically all the results given in this paper, I found that a good deal had been already done by others, though in a somewhat different way. In particular, the general problem of Prop. 3 above is completely solved in a paper by Robert Brodie, published in the T.R.S.E., Vol. XXXVI. part 2, p. 307, entitled "On Professor Kelland's Problem on Superposition," in which he refers to some previous papers by Kelland in earlier volumes. Again, Perigal (*Messenger of Mathematics*, II. p. 103) gives a solution for the case of *Euclid* I. 47, which is generalised to some extent by Harry Hart (*Messenger*, VI. p. 150).

On the Elementary Differentiations.

By R. F. Muirhead, M.A., B.Sc.

[redacted]

Fourth Meeting, February 14th, 1896.

Dr PEDDIE, President, in the Chair.

Note on a Certain Harmonical Progression.
Note on Continued Fractions.
On Methods of Election.

BY PROFESSOR STEGGALL.

A Simple Method of Finding any Number of Square Numbers whose Sum is a Square.

BY ARTEMAS MARTIN, LL.D.

I.—Take the well-known identity

$$(w+z)^2 = w^2 + 2wz + z^2 = (w-z)^2 + 4wz \quad - \quad - \quad (1).$$

Now if we can transform $4wz$ into a square we shall have *two* square numbers whose sum is a square. This will be effected by taking $w = p^2$, $z = q^2$, for then $4wz = 4p^2q^2 = (2pq)^2$ and we have

$$(p^2+q^2)^2 = (p^2-q^2)^2 + (2pq)^2 \quad - \quad - \quad - \quad (2).$$

See *Mathematical Magazine*, Vol. II., No. 5, p. 69.

In (2) the values of p and q may be chosen at pleasure, but to have numbers that are prime to each other p and q must also be prime to each other and one odd and the other even.

Examples.—1. Take $p = 2$, $q = 1$; then we find

$$3^2 + 4^2 = 5^2.$$

2. Take $p = 3$, $q = 2$; then we shall have

$$5^2 + 12^2 = 13^2.$$

3. Take $p = 4$, $q = 1$; then we get

$$8^2 + 15^2 = 17^2.$$

And so on, *ad lib.*

II.—We can obtain from (1) any number of squares whose sum is a square by simply substituting for w the sum of two, of three, of four, etc., other quantities.

In (1) put $x+y$ for w and we have

$$(x+y+z)^2=(x+y-z)^2+4(x+y)z,$$
$$=(x+y-z)^2+4xz+4yz,$$
$$=(x+z-y)^2+4xy+4yz,$$
$$=(y+z-x)^2+4xy+4xz.$$

Assume $x=p^2$, $y=q^2$, $z=r^2$, and we have

$$(p^2+q^2+r^2)^2=(p^2+q^2-r^2)^2+(2pr)^2+(2qr)^2,$$
$$=(p^2+r^2-q^2)^2+(2pq)^2+(2qr)^2,$$
$$=(q^2+r^2-p^2)^2+(2pr)^2+(2pq)^2. \quad - \quad (3),$$

three sets of *three* squares, the sum of each of which is $(p^2+q^2+r^2)^2$, where p, q, r may have any integer values.

See *Mathematical Magazine*, Vol. II., No. 5, p. 72.

Examples.—1. Take $p=4$, $q=2$, $r=1$; then we have

$$21^2=19^2+8^2+4^2=16^2+13^2+4^2=16^2+11^2+8^2.$$

2. Take $p=5$, $q=3$, $r=1$, and we get

[redacted]

Now take $v=p^2$, $x=q^2$, $y=r^2$, $z=s^2$, and we have

$$\begin{aligned}(p^2+q^2+r^2+s^2)^2 &= (p^2+q^2+r^2-s^2)^2+(2ps)^2+(2qs)^2+(2rs)^2,\\ &= (p^2+q^2+s^2-r^2)^2+(2pr)^2+(2qr)^2+(2rs)^2,\\ &= (p^2+r^2+s^2-q^2)^2+(2pq)^2+(2qr)^2+(2qs)^2,\\ &= (q^2+r^2+s^2-p^2)^2+(2pq)^2+(2pr)^2+(2ps)^2. \quad (4),\end{aligned}$$

four sets of *four* square numbers, the sum of each set being $(p^2+q^2+r^2+s^2)^2$, where p, q, r, s may have any integer values.

Examples.—1. Take $p=5$, $q=3$, $r=2$, $s=1$, and we find

$$\begin{aligned}39^2 &= 37^2+10^2+6^2+4^2=31^2+20^2+12^2+4^2\\ &= 30^2+21^2+12^2+6^2=30^2+20^2+11^2+10^2.\end{aligned}$$

2. Take $p=5$, $q=4$, $r=3$, $s=1$; then we shall have

$$\begin{aligned}51^2 &= 49^2+10^2-8^2+6^2=33^2+30^2+24^2+6^2\\ &= 40^2+24^2+19^2+8^2=40^2+30^2+10^2+1^2\end{aligned}$$

3. Take $p=7$, $q=5$, $r=2$, $s=1$, and we will get

$$\begin{aligned}79^2 &= 77^2+14^2+10^2+4^2=71^2+28^2+20^2+4^2\\ &= 70^2+29^2+20^2+10^2=70^2+28^2+19^2+14^2.\end{aligned}$$

And so on, *ad lib.*

IV.—In the same way we might find formulas for five squares whose sum is a square, *six* squares whose sum is a square, and so on; but from what has been done above it is obvious that we may write at once

$$\begin{aligned}(a^2_1+a^2_2+a^2_3+\ldots+a^2_n)^2 &= (a^2_1+a^2_2+a^2_3+\ldots+a^2_{n-1}-a^2_n)^2+(2a_1a_n)^2\\ &\quad +(2a_2a_n)^2+(2a_3a_n)^2+\ldots\ldots+(2a_{n-1}a_n)^2 \quad - \quad - \quad (5),\end{aligned}$$

one set of n square numbers whose sum is a square; and we can obtain by cyclic permutation $(n-1)$ other sets, the sum of which is equal to $(a^2_1+a^2_2+a^2_3+\ldots\ldots+a^2_n)^2$, where a_1, a_2, $a_3\ldots\ldots a_n$ may have any integer values chosen at pleasure.

Properties of some Groups of Wallace Lines.

BY R. TUCKER, M.A.

1. Take the point P in the arc AB, and let $\angle$ ABP $= \theta$, then the trilinear equation of the Wallace line of P (p say) is

$$a\alpha\cos(C-\theta)\cos(B+\theta)\sin\theta - b\beta\cos\theta\sin(C-\theta)\cos(B+\theta)$$
$$+ c\gamma\cos(C-\theta)\sin(B-\theta)\cos\theta = 0 \qquad \text{(A)}$$

2. (*a*) *The lines AP, BQ, CR drawn parallel to the sides BC, CA, AB of the triangle ABC.*

In this case p, q, r, by [§ 17] * are readily seen to be concurrent. To find the point, we must solve the equations

for P($\theta = C - B$), for Q($\theta = A$),

$$a\alpha\cos B\cos C\sin(B-C) - b\beta\cos(B-C)\sin B\cos C$$
$$+ c\gamma\cos B\sin C\cos(B-C) = 0,$$

$$-a\alpha\cos(C-A)\cos A\sin A + b\beta\cos A\sin(C-A)\cos C$$
$$+ c\gamma\cos(C-A)\sin C\cos A = 0.$$

The point is given by

$$\alpha/\cos^2 A\cos(B-C) = \beta/\cos^2 B\cos(C-A) = \gamma/\cos^2 C\cos(A-B). \dagger$$

3. (*b*) *PQ drawn parallel to AB.*

Then P(θ), Q($C-\theta$) have for Wallace lines, viz., p, equation (A),

[redacted]

4. Take $AP = BQ = BR = CS = CT = AV$ and suppose they subtend the angle θ at the circumference and let the Wallace lines of P, Q, R, S, T, V, *i.e.*, p, q, r, s, t, v intersect in a', b', c', d', e', f', then [§ 17] angles at

$a', c', e' = 2\theta$, acute angles at $b' = C - 2\theta$, at $d' = A - 2\theta$, and at $f'B - 2\theta$.

Then r, s ; q, t intersect on $\perp^r$ from A,

t, v ; p, s " " " " B,

and p, q ; r, v " " " " C ;

and the intercepts on the respective perpendiculars are

$$a\sin 2\theta, \qquad b\sin 2\theta, \qquad c\sin 2\theta.$$

5. I get these results from the equations referred to BC, BA as axes. They are

$$p. \quad y\cos(B+\theta) + x\cos\theta = 2R\cos\theta\cos(B+\theta)\sin(C-\theta) \qquad \text{(i.)}$$

$$q. \quad y\cos(A+\theta) - x\cos(C-\theta) = 2R\cos(C-\theta)\cos(A+\theta)\sin\theta \qquad \text{(ii.)}$$

$$r. \quad -y\cos(A-\theta) + x\cos(C+\theta) = 2R\cos(C+\theta)\cos(A-\theta)\sin\theta \qquad \text{(iii.)}$$

$$s. \quad y\cos\theta + x\cos(B+\theta) = 2R\cos(B+\theta)\cos\theta\sin(A-\theta) \qquad \text{(iv.)}$$

t. and v. may be similarly obtained.

p. and s. intersect in $y\sin B = 2R\cos\theta\cos(B+\theta)\cos C$,

$$x\sin B = 2R\cos\theta\cos(B+\theta)\cos A,$$

hence they intersect on the $\perp^r$ from B at m (say),

i.e. $\perp^r$ from m on BC is $2R\cos\theta\cos(B+\theta)\cos C$,

hence $$mE = 2R[\sin A\sin C - \cos\theta\cos(B+\theta)].$$

6. Again r, s intersect in d' so that

$$d'D = 2R\cos(B+\theta)\cos(C+\theta) ;$$

hence $$d'D\cos(A+\theta) = f'E\cos(B+\theta) = b'F\cos(C+\theta),$$

and also mf'

$= 2R[\sin A\sin C - \cos\theta\cos(B+\theta) - \cos(C+\theta)\cos(A+\theta)]$

$= R[\cos(C-A) + \cos B - \cos B - \cos(B+2\theta) - \cos(C+A+2\theta) - \cos(C-A)]$

$= b\sin 2\theta.$

Hence $$ld' : mf' : nb' = a : b : c.$$

Again $\qquad mH = mE - EH = 2R\sin\theta\sin(B+\theta)$,

i.e. $\quad Hl : Hm : Hn = \sin(A+\theta) : \sin(B+\theta) : \sin(C+\theta)$;

and $\qquad Hf' = 2R\sin\theta\sin(B-\theta)$,

$$\therefore Hd' : Hf' : Hb' = \sin(A-\theta) : \sin(B-\theta) : \sin(C-\theta).$$

7. By the cosine method we readily get

$$b'm = a\sin\theta \text{ and } \therefore b'm : d'n : f'l = a : b : c;$$

also $\qquad b'l = b\sin\theta$ and $\therefore f'n : b'l : d'm = a : b : c$.

The areas of $mb'lf'$ and $md'nf'$ are respectively

$$R^2\sin^2\theta[2\sin A\sin B\sin C(1+2\cos2\theta) - (\cos2C - \cos2A)\sin2\theta]$$

and $R^2\sin^2\theta[2\sin A\sin B\sin C(1+2\cos2\theta) + (\cos2C - \cos2A)\sin2\theta]$,

hence area of hexagon $\qquad = 4\sin A\sin B\sin C R^2\sin^2\theta(1+2\cos2\theta)$.

8. We may note that the angles are

$$\left.\begin{aligned} Hb'l &= Hf'l = A+\theta, \\ Hb'm &= Hd'm = B+\theta, \\ Hd'n &= Hf'n = C+\theta, \end{aligned}\right\} \text{ and } \left.\begin{aligned} Hmd' &= Hnd' = A-\theta, \\ Hlf' &= Hnf' = B-\theta, \\ Hlb' &= Hmb' = C-\theta. \end{aligned}\right\}$$

9. The intersection of p, v, *i.e.*, a' lies on the line

$$y/b\cos(B-C) + x/b\cos C = 1,$$

which cuts BC at a distance from B, towards C, $=$ CD, and so for the analogous lines.

[redacted]

Again since $\angle bed =$ right angle $= 2^{ce} \angle bcd,$

and ac is a diameter, $\therefore e$ is centre of circle $abcd$.

Hence $eg, em \parallel$ to $bc, cd.$

We have $aN^2 + cN^2 = 2Ne^2 + 2ea^2 = 6\rho^2$

11. Let ab, bc, cd, da be represented by a', b', c', d' respectively, and take $\angle cad = \phi,$

then $c' = 2\sqrt{2}\rho\sin\phi, \qquad d' = 2\sqrt{2}\rho\cos\phi,$

$$a' = 2\rho(\sin\phi - \cos\phi), \quad b' = 2\rho(\sin\phi + \cos\phi).$$

From the figure we see $a'^2 + b'^2 = ac^2 = c'^2 + d'^2$,

and rectangle $bgdl = \frac{1}{2}a'b'$, rectangle $bfdm = \frac{1}{2}c'd'$,

$\therefore$ sum of rectangles $= abcd$;

$$aN^2 = d'^2 + \rho^2 - 2d'\rho\cos\left(\phi - \frac{\pi}{4}\right) = \rho^2[3 + 2(\cos 2\phi - \sin 2\phi)],$$

$$cN^2 = \rho^2[3 - 2(\cos 2\phi - \sin 2\phi)].$$

12. Let h be the mid point of the third diagonal kn,

then $bh = \frac{1}{2}kn = dh,$

$$= \tfrac{1}{2}\sqrt{a'^2 + b'^2} = ae = \rho\sqrt{2},$$

hence, since we know that eNh is a straight line, h is on the Nine-point circle.

13. PQRSTV is a regular six-side in the circle and the Wallace lines are indicated by p', q', r', s', t', v' respectively.

The angles [§ 17] at f, b, d are each 30°,

and at a, c, e, are supplements of 30°.

By [§ 19] p', s'; q', t'; r', v' intersect at right angles in g, h, k respectively, on the nine-point circle.

From the symmetry of the figure ghk is an equilateral triangle, or we may prove this by finding the projections of gh on BC and at right angles thereto.

The projections will be found to be respectively

$$\frac{R\sqrt{3}}{2}\cos\left(B - C + 4\theta + \frac{\pi}{3}\right), \quad \frac{R\sqrt{3}}{2}\sin\left(B - C + 4\theta + \frac{\pi}{3}\right),$$

hence $gh = R\sqrt{3}/2 = \rho\sqrt{3}.$

14. Since $\angle kqh = \angle kgh = 60°$, $\therefore gh = qb = qe$, and $qh \parallel fg$. Now $\angle clh = \angle gkh$ $\therefore$ l is mid point of wd, and similarly m of bz, and n of fx.
The $\angle cfg = \angle ckg = \angle qhg = \angle hgf$,

$$\therefore \ cf \text{ is bisected in } h \text{ and } = 2\rho\sqrt{3}.$$

Also q is mid point of ct.

15. It is seen that the arcs ph, qg, rk are equal, and if we suppose the $\angle$ they subtend at the circumference to be ϕ,

then $$wd = 2lh = 4\rho\cos\phi = xf = zb.$$

Since hq bisects $\angle chg$ and is $\parallel hg$

$$\therefore \ ch = gh \text{ and } cw = gl,$$

hence $$cd = ef = ab \text{ and } bc = de = af$$

$$\therefore \ bdf \text{ is equilateral, as also is } stv.$$

[Note.—The reader is requested to draw the figures. The following details, which refer to the figures which accompanied the paper, will render the text more intelligible :

In § 4, A = 60°, B = 45°, C = 75°, $\theta = 18°$.

vp meet in a'
pq „ „ b'
and so on to tv in f'. } angle $m = 100°$,

In § 10, A = 61°, B = 39°, C = 80°, $\theta = 20°$,

[redacted]

Fifth Meeting, March 13th, 1896.

Dr PEDDIE, President, in the Chair.

On Curve Tracings.

BY G. DUTHIE, M.A.

Note on Four-Dimensional Figures.

BY J. D. HÖPPNER.

By assuming that multiplication by a line is the true operation corresponding to the passing from space of n dimensions to space of $n+1$ dimensions we may arrive very simply at certain well-known results in geometry of higher dimensions.

A finite straight line may be symbolised by

$$1.a^1 + 2a^0$$

which indicates that the line consists of one line quantity *and* (+) is fixed by two non-dimensional quantities. For simplicity, we may write the above symbol in the form

$$a+2.$$

The algebraic square of this quantity is

$$a^2 + 4a + 4$$

and this is also the symbol of a geometrical square, having 1 area bounded by 4 sides and 4 points. Raising $(a+2)$ to the 3rd power we obtain

$$a^3 + 6a^2 + 12a + 8$$

which represents 1 volume, bounded by 6 faces, 12 lines, and 8 points.

Passing on to a higher dimension, we obtain as the symbol of the four-dimensional figure

$$(a+2)^4 = a^4 + 8a^3 + 24a^2 + 32a + 16$$

consisting of 1 four-dimensional region bounded by 8 cubes, 24 squares, 32 lines, and 16 points.

The method admits of other applications and of obvious extension to higher dimensions.

Sixth Meeting, May 8th, 1896.

Dr PEDDIE, President, in the Chair.

Note on the Formula for tan (A+B).

By Professor STEGGALL.

FIGURE 43.

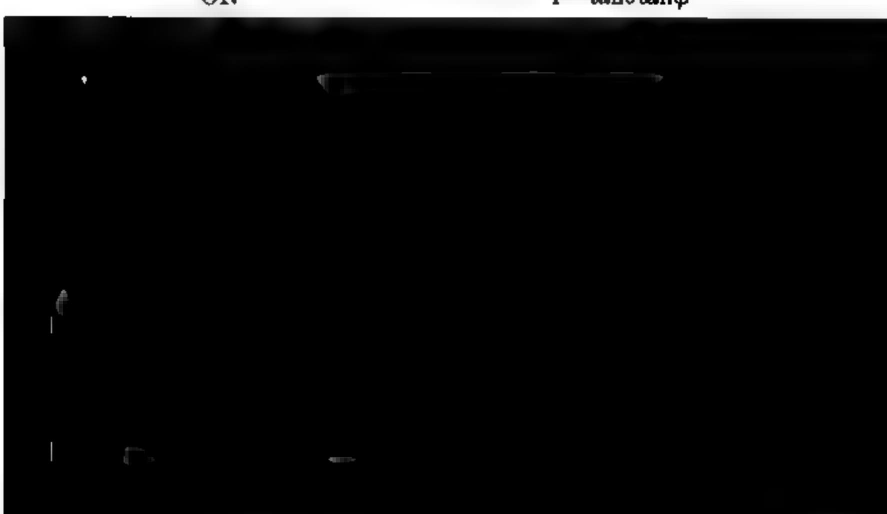
From the figure we have at once

$$\tan\theta + \tan\phi = \frac{BC}{AE} = \frac{2BN}{AE}$$

$$\tan\theta\tan\phi = \frac{BE}{AE}\cdot\frac{ED}{BE} = \frac{ED}{AE}$$

$$1 - \tan\theta\tan\phi = \frac{2ON}{AE}$$

$$\therefore \frac{BN}{ON} = \tan(\theta+\phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi};$$



also known that this process can be extended. I propose to examine various results connected with these lines.

Let the angular coordinate of any points on a circle be 2α, 2β, 2γ, . . the radius a, and let any other point have an angular coordinate $\pi - 2\theta$: the chord (α, β) is

$$x\cos(\alpha+\beta) + y\sin(\alpha+\beta) = a\cos(\alpha-\beta)$$

the axes being through the centre as usual. This equation may be written

$$(x + a\cos 2\theta)\cos(\alpha+\beta) + (y - a\sin 2\theta)\sin(\alpha+\beta)$$
$$= 2a\cos(\alpha+\theta)\cos(\beta+\theta) \qquad - \quad - \quad - \quad (1)$$

The perpendicular from the point $(\pi - 2\theta)$ has for equation

$$(x + a\cos 2\theta)\sin(\alpha+\beta) - (y - a\sin 2\theta)\cos(\alpha+\beta) = 0 \qquad - \quad (2)$$

and the line

$$(x + a\cos 2\theta)\cos(\alpha+\beta+\gamma+\theta) + (y - a\sin 2\theta)\sin(\alpha+\beta+\gamma+\theta)$$
$$= 2a\cos(\alpha+\theta)\cos(\beta+\theta)\cos(\gamma+\theta) \qquad - \quad - \quad (3)$$

passes through the intersection of (1) and (2). But its symmetry at once shows that it is the pedal line of the triangle $(2\alpha, 2\beta, 2\gamma)$.

Proceeding to a fourth point

$$(x + a\cos 2\theta)\cos(\alpha+\beta+\gamma+\delta+2\theta)$$
$$+ (y - a\sin 2\theta)\sin(\alpha+\beta+\gamma+\delta+2\theta)$$
$$= 2a\cos(\alpha+\theta)\cos(\beta+\theta)\cos(\gamma+\theta)\cos(\delta+\theta) \quad - \quad (4)$$

is the Simson line of the quadrangle 2α, 2β, 2γ, 2δ, and so on.

If we choose the original axis of x to coincide in direction with the mean of the angular directions 2α, 2β,

$$\alpha + \beta + \gamma + \quad . \quad . \quad = 0$$

and our line becomes in the general case

$$(x + a\cos 2\theta)\cos(n-2)\theta + (y - a\sin 2\theta)\sin(n-2)\theta$$
$$= 2a\cos(\alpha+\theta)\cos(\beta+\theta) \quad . \quad . \quad .$$

or

$$x\cos(n-2)\theta + y\sin(n-2)\theta$$
$$= 2a\cos(\alpha+\theta)\cos(\beta+\theta) \quad . \quad . \quad - a\cos n\theta$$
$$= a\left(\frac{\Sigma\cos(\overline{\alpha+\theta}\pm\overline{\beta+\theta}\pm\overline{\gamma+\theta})}{2^{n-2}} - \cos n\theta\right) \qquad - \quad - \quad (5)$$

Now in the case of a triangle, if θ varies, this gives

$$x\cos\theta + y\sin\theta = a\left(\frac{\cos(\theta - 2\alpha) + \ . \ . \ - \cos 3\theta}{2}\right)$$

or
$$\left(x - \frac{\Sigma\cos 2\alpha}{2}\right)\cos\theta + \left(y - \frac{\Sigma\sin 2\alpha}{2}\right) = -\frac{a\cos 3\theta}{2}$$

a line enveloping a three-cusped hypocycloid whose centre is at the nine-points centre . . . . (6)

In the case of a quadrangle

$$\begin{aligned} x\cos 2\theta + y\sin 2\theta = {} & a\frac{\cos 2(\alpha+\beta) + \quad . \quad .}{4} \\ & + a\frac{\cos(2\theta - 2\alpha) + \quad . \quad .}{4} \\ & + \qquad \frac{3a\cos 4\theta}{4} \end{aligned}$$

which only envelopes a four-cusped hypocycloid if

$$\cos 2(\alpha+\beta) + \quad . \quad . \quad + \quad . \quad . \quad = 0.$$

If we consider the given point as fixed, we may take $\theta = 0$, and change the axes so that it is at the origin. The Simson line now becomes from (4)

$$x\cos(\alpha+\beta+\gamma+ \quad . \quad . \quad) + y\sin(\alpha+\beta+\gamma+ \quad . \quad)$$
$$= 2a\cos\alpha\cos\beta\cos\gamma\cos\delta \quad . \quad . \quad .$$

and if 2θ be the inclination of the mean line and $2\alpha'$, $2\beta'$, 2γ, . .

[redacted]

For a quadrangle

$$x\cos 4\theta + y\sin 4\theta = \frac{a}{4}\{\cos 2(\alpha+\beta) + \ldots \ldots$$
$$+ \cos 2(\theta - \alpha) + \ldots$$
$$+ \cos 4\theta\}.$$

In the case of regular figures these results are much simplified, and it will be found best to start *de novo* from the equation (4): we may take the axis of x to pass through a corner when n is odd, and through a mid point of a side if n is even: so that

$$n \text{ odd}, \qquad \alpha = \frac{0}{n}, \qquad \beta = \frac{2\pi}{n}, \qquad \gamma = \frac{2\pi}{n} \quad \ldots$$

$$n \text{ even}, \qquad \alpha = \frac{\pi}{2n}, \qquad \beta = \frac{3\pi}{2n}, \qquad \gamma = \frac{5\pi}{2n} \quad \ldots$$

The lines are respectively

$$x\cos\left(\frac{n-1}{2}\pi + (n-2)\theta\right) + y\sin\left(\frac{n-1}{2}\pi + (n-2)\theta\right)$$
$$= -a\cos\left(\frac{n-1}{2}\pi + n\theta\right) + 2a\cos\theta\cos\left(\frac{\pi}{n} + \theta\right) \quad \ldots \ldots$$

and

$$x\cos\left(\frac{n\pi}{2} + (n-2)\theta\right) + y\sin\left(\frac{n\pi}{2} + (n-2)\theta\right)$$
$$= -a\cos\left(\frac{n\pi}{2} + n\theta\right) + 2a\cos\left(\frac{\pi}{2n} + \theta\right)\cos\left(\frac{3\pi}{2n} + \theta\right) \quad \ldots \ldots$$

But by a known trigonometrical formula

$$\cos\theta\cos\left(\frac{\pi}{n} + \theta\right)\cos\left(\frac{2\pi}{n} + \theta\right) \quad \ldots \quad n \text{ factors (odd)}$$
$$= (-1)^{\frac{n-1}{2}} \cdot \frac{\cos n\theta}{2^{n-1}}$$

$$\cos\left(\theta + \frac{\pi}{2n}\right)\cos\left(\theta + \frac{3\pi}{2n}\right) \quad \ldots \quad = \frac{\cos n\left(\theta + \frac{\pi}{2}\right)}{2^{n-1}}$$

whence the line, in either case, becomes

$$x\cos(n-2)\theta + y\sin(n-2)\theta$$
$$= a\cos n\theta \cdot \frac{1 - 2^{n-2}}{2^{n-2}} \quad - \quad - \quad - \qquad (8)$$

which envelopes an n-cusped hypocycloid.

In the next place, if the point is fixed, transfer to it as orgin and we get as in equation (7)

$$x\cos n\theta + y\sin n\theta = 2a\cos(\alpha+\theta)\cos(\beta+\theta) \quad . \quad .$$
$$= \frac{a\cos\theta}{2^{n-2}}$$

which passes through the fixed point $x = a/2^{n-2}$, $y = 0$

Hence the Simson line of a regular polygon with respect to a moving point envelopes a hypocycloid and the Simson line with respect to a fixed point of a regular polygon that slides round a circle passes through another fixed point.

On Deducing the Properties of the Trigonometrical Functions from their Addition Equations.

By R. F. Muirhead, M.A., B.Sc.

1. Take first the Addition-Formula of the Tangent:

$$\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y} \quad - \quad - \quad - \quad \text{I.}$$

Take $y=0$, $\quad \therefore \tan x=\dfrac{\tan x+\tan 0}{1-\tan x\tan 0}$ - - - (1)

Assume that $1-\tan x\tan 0$ is not zero for all value of x - - II.

and we have $\quad \tan x(1-\tan x\tan 0)=\tan x+\tan 0$

$$\therefore \tan 0(1+\tan^2 x)=0 \quad - \quad - \quad - \quad (2)$$

Assume that there is *one* value at least for x for which

$$\tan^2 x \neq -1 \quad \text{and } 1-\tan x\tan 0\neq 0 \quad - \quad - \text{III.}$$

and we have $\quad \tan 0=0$ - - - - - - (3)

Hence $\quad 0=\tan(x-x)=\dfrac{\tan x+\tan(-x)}{1+\tan^2 x}$

$$\therefore \tan x=\tan(-x) \quad - \quad - \quad - \quad - \quad (4)$$

hence, writing $-y$ for y in I.

$$\tan(x-y)=\frac{\tan x-\tan y}{1+\tan x\tan y} \quad - \quad - \quad - \quad (5)$$

Again $\quad \tan(x+h)-\tan x=\tan h(1+\tan^2 x)\div(1-\tan x\tan h)$ - (6)

By taking h small enough, the denominator may be made positive, so long as $\tan x$ is finite. Hence $\tan(x+h)>\tan x$ and as x increases from 0, $\tan x$ increases at an increasing rate, and must eventually become $=1$

Let $\dfrac{R}{2}$ be the value of x which makes $\tan\dfrac{R}{2}=1$ - - - IV.

Then $\quad \tan R=\tan\left(\dfrac{R}{2}+\dfrac{R}{2}\right)=\dfrac{1+1}{1-1}=\infty$

$$\therefore \tan 2R=\frac{2\tan R}{1-\tan^2 R}=2\div\left(\frac{1}{\tan R}-\tan R\right)=0$$

Hence $\quad \tan(2R+x)=\dfrac{0+\tan x}{1+0}=\tan x$ - - - (7)

Thus $\tan x$ is periodic, and the period is $2R$.

Again $\qquad \tan(2\mathrm{R} - x) = \tan(-x) = -\tan x.$

And $\qquad \tan(\mathrm{R} - x) = \dfrac{\tan\mathrm{R} - \tan x}{1 + \tan\mathrm{R}\tan x} = \dfrac{1}{\tan x}$

Hence the graph of $\tan x$ might be roughly sketched.

Again $\qquad 1 = \tan\dfrac{\mathrm{R}}{2} = \tan\left(\dfrac{\mathrm{R}}{4} + \dfrac{\mathrm{R}}{4}\right) = \dfrac{2\tan\dfrac{\mathrm{R}}{4}}{1 - \tan^2\dfrac{\mathrm{R}}{4}}$

Hence $\qquad \tan\dfrac{\mathrm{R}}{4} = \pm\sqrt{2} - 1 \quad \therefore \tan\dfrac{\mathrm{R}}{4} = \sqrt{2} - 1$

In this manner we can calculate $\tan\dfrac{\mathrm{R}}{2^n}$ where n is any integer. Hence by the Addition Formula we can find $\tan\dfrac{m\mathrm{R}}{2^n}$ where m is any integer, and in this way we can approximate to the value of the tangent of any angle between 0 and R; and then, using (4) and (7) to that of any angle whatever.

R depends on the unit of angular measurement; or *vice versa*, if R be arbitrarily chosen, the unit of measurement depends on it.

If we take the limit when $x = 0$ of $\tan x/x$ to be $= 1$ as in radian measure, we get from (6)

[redacted]

Let $\phi(x)$ and $\psi(x)$ be two functions of x such that

$$\phi(x+y)=\phi(x)\psi(y)+\psi(x)\phi(y) \quad - \quad - \quad - \quad \text{I.}$$

$$\psi(x+y)=\psi(x)\psi(y)-\phi(x)\phi(y) \quad - \quad - \quad - \quad \text{II.}$$

so that ϕ and ψ are functions having the same Addition-Formulae as the sine and the cosine respectively; we shall seek to deduce from I. and II. the nature of ϕ and ψ, pointing out as we proceed, any further assumptions that we make as to the nature of ϕ and ψ.

In I. put $y=0 \quad \therefore \quad \phi(x)=\phi(x)\psi(0)+\psi(x)\phi(0)$

$$\therefore \quad \phi x(1-\psi(0))=\psi(x)\phi(0) \quad - \quad - \quad (1)$$

Similarly from II. $\quad \psi(x)(1-\psi(0))=-\phi(x)\phi(0) \quad - \quad - \quad - \quad (2)$

Multiplying across and transposing,

$$[\{\psi(x)\}^2+\{\phi(x)\}^2\{1-\psi(0)\}\phi(0)=0 \quad - \quad - \quad (3)$$

Assume that $(\phi x)^2+(\psi x)^2$ is not $=0$ for all values of x - - III.

This is certainly true if $\phi(x)$ and $\psi(x)$ are real and not both always $=0$

Hence either $\qquad \phi(0)=0$ or $1-\psi(0)=0$

Now if $\phi(0)=0$ then by (1) $\psi(0)=1$ unless $\phi(x)=0$ for all values of x or $\psi(x)=\infty$ for all values of x.

And by (2) $\psi(0)=1$ unless $\psi(x)=0$ for all values of x or $\phi(x)=\infty$ for all values of x.

On the whole then, the alternative $\phi(0)=0$ involves also that $\psi(0)=1$ if we assume:

There is some value of x for which ϕ or ψ is neither ∞ nor 0. - - - - - - - - - - IV.

And it is clear that, on this assumption, the equation $\phi(0)=0$ can be deduced from the other alternative $\psi(0)=1$. Hence finally

$$\left.\begin{array}{l}\phi(0)=0\\ \psi(0)=1\end{array}\right\} \quad - \quad - \quad - \quad - \quad - \quad (4)$$

Put $y=-x$ in I. and II. and we have

$$0=\phi(0)=\phi(x-x)=\phi(x)\psi(-x)+\psi(x)\phi(-x) \quad - \quad (5)$$

$$1=\psi(0)=\psi(x-x)=\psi(x)\psi(-x)-\phi(x)\phi(-x) \quad - \quad (6)$$

Eliminating successively $\psi(-x)$ and $\phi(-x)$ we have

$$\phi(x) = \{(\phi x)^2 + (\psi x)^2\}\phi(-x) \quad - \quad - \quad - \quad (7)$$

$$\psi(x) = \{(\phi x)^2 + (\psi x)^2\}\psi(-x) \quad - \quad - \quad - \quad (8)$$

Assume $$(\psi x)^2 + (\phi x)^2 \equiv 1 \quad - \quad - \quad - \quad - \quad \text{V.}$$

(Note that this includes Assumption III.)

Hence by (7) and (8), $$\phi(x) = -\phi(-x) \quad - \quad - \quad - \quad - \quad (9)$$

$$\psi(x) = \quad \psi(-x) \quad - \quad - \quad - \quad - \quad (10)$$

Hence from I. and II., by putting $-y$ for y, we have the Subtraction Formulae:

$$\left.\begin{aligned} \phi(x-y) &= \phi x \psi y - \psi x \phi y \\ \psi(x-y) &= \psi x \psi y + \phi x \phi y \end{aligned}\right\} \quad - \quad - \quad - \quad (11)$$

It is clear that at this stage V., (9), (10), and (11) are all equivalent, so that any one of them being assumed, the others would follow.

As in books on Elementary Trigonometry we can now deduce from I., II., and (11) all the formulae for functions of multiples or submultiples of x, and such formulae as

$$\phi(x) + \phi(y) = 2\phi\left(\frac{x+y}{2}\right)\psi\left(\frac{x-y}{2}\right).$$

Assume now that $\phi(x)$ is not always $=0$ and is real and continuous, and that it becomes positive* at first as x increases

[redacted]

$$\left.\begin{array}{llll}
\text{Then } \psi(R-x) & = \psi R\psi x + \phi R\phi x & & \\
 & = \phi x & & \\
\text{Similarly } \phi(R-x) = & \psi x & & \\
\quad\text{,,}\quad \phi(2R-x) = & \phi(x) & & \\
\quad\text{,,}\quad \psi(2R-x) = & -\psi(x) & & \\
\quad\text{,,}\quad 0 & = \phi(2R) = \phi(4R) = & \phi(6R) = & \\
\quad\text{,,}\quad 0 & = \psi(R) = \psi(3R) = & \psi(5R) = & \\
\quad\text{,,}\quad 1 & = -\phi(3R) = \phi(5R) = & -\phi(7R) = & \\
\quad\text{,,}\quad 1 & = -\psi(2R) = \psi(4R) = & -\psi(6R) = &
\end{array}\right\} \quad (14)$$

Hence $\phi(4R+x) = \phi x$ and $\psi(4R+x) = \psi x$ - - - - (15)

Thus ϕ and ψ are both periodic functions, the period being $=4R$, and it is clear that if the values of ϕ from $x=0$ to $x=R$ were tabulated, we could at once find the value of ϕ or ψ for any other x by means of (14) and (15).

We are in a position to trace the ϕ and ψ curves roughly, and they are obviously similar to $\sin x$ and $\cos x$, the unit angle being $\frac{1}{R}$ of a right angle.

The actual values of ϕ and ψ for as many values of x as we please can easily be calculated, *e.g.*,

$$\frac{R}{2} = R - \frac{R}{2} \quad \therefore \text{ by (14) } \phi\left(\frac{R}{2}\right) = \psi\left(\frac{R}{2}\right)$$

Hence by V. $\psi\left(\frac{R}{2}\right) = \pm\frac{1}{\sqrt{2}}$, and we must take the upper sign since ψ is positive till $x = R$.

Hence also $\phi\left(\frac{R}{2}\right) = \frac{1}{\sqrt{2}}$ - - - - - - - - (16)

By repeated application of the formula

$$\psi\left(\frac{x}{2}\right) = \sqrt{\left(\frac{1+\psi x}{2}\right)}$$

we can get the value of $\psi(x)$ for as small a value of x as we please: thence finding $\phi(x)$ for the same value of x by means of V., we could, by the Addition-Formulae, interpolate as many values as we pleased of ϕ and ψ between the values found, of which (16) is a specimen.

Thus we have shown that the two functions defined by I., II., III., IV., V., VI. are one-valued.

Since the sine and cosine satisfy these assumptions, ϕ and ψ must be identical with them.

If we suppose the sine and cosine defined geometrically, then to prove their identity with the functions ϕ and ψ as above conditioned, it would only be necessary to prove geometrically the Addition Equations and the Subtraction Equations and verify the assumptions III., IV., VI.

R is arbitrary, and its choice determines the unit of angular measurement. To connect R with the limiting value of $\phi(x)/x$ we might use the formula

$$1=\phi(\mathrm{R})=2^n\psi\left(\frac{\mathrm{R}}{2}\right)\psi\left(\frac{\mathrm{R}}{2^2}\right)\ \cdots\ \psi\left(\frac{\mathrm{R}}{2^n}\right)\phi\left(\frac{\mathrm{R}}{2^n}\right)$$

$$\therefore\ 1=\sqrt{\frac{1+0}{2}}\cdot\sqrt{\frac{1+\frac{1}{\sqrt{2}}}{2}}\cdot\sqrt{\frac{1+\sqrt{\left(\frac{1}{2}+\frac{1}{2\sqrt{2}}\right)}}{2}}\ \cdots\ \times\mathrm{R}\frac{\phi\left(\frac{\mathrm{R}}{2^n}\right)}{\frac{\mathrm{R}}{2^n}}$$

$\therefore\ \frac{1}{\mathrm{R}}=$ a certain number $\times k$, where k is the limiting value when $x=0$ of $\phi(x)/x$. The number referred to is of course $\frac{2}{\pi}$. Instead of choosing R arbitrarily we might choose k, and so fix R.

3. In order to prove that the assumptions III., IV., V., VI. are all *necessary* we should have to show that if any one is omitted, the functions ϕ and ψ may be different from the sine and cosine.

[redacted]

Thus the functions ϕ_1 and ψ_1 satisfy the condition $\phi_1^2+\psi_1^2=1$, and are therefore, as we saw, identical with the sine and cosine of x.

To find the nature of the more general functions ϕ and ψ let us put
$$\phi(x)\equiv f(x)\phi_1(x)$$
then by (17) we get
$$\psi(x)=f(x)\psi_1(x)$$

Substituting in the Addition Formulae we get
$$f(x+y)=f(x)f(y)$$

Hence, as in the proof of the Exponential Theorem,

$f(x)\equiv a^x$, where a is an arbitrary constant.

Thus the most general functions conditioned by I., II., III., IV., VI. are $a^x\sin x$ and $a^x\cos x$. The special case $a=0$ giving $\sin x$ and $\cos x$ is got by introducing either V. or any of its equivalents.

If we drop Condition VI. as well as V., *i.e.*, if we admit imaginaries, the present mode of treatment becomes inconvenient.

In VI. the assumption of continuity is required by the occurrence of $\phi\left(x+\frac{h}{2}\right)$ and $\psi\left(x+\frac{h}{2}\right)$ on the right hand sides of equations (12). By writing (12) in the form
$$\phi(x+h)-\phi(x)=2\phi\left(\frac{h}{2}\right)\left\{\psi(x)\psi\left(\frac{h}{2}\right)-\phi(x)\phi\left(\frac{h}{2}\right)\right\} \text{ etc.,}$$
we see that if $\phi\left(\frac{h}{2}\right)\div\frac{h}{2}$ has a finite limit, the functions are continuous, *i.e.*, if $\phi(x)$ *begins* by being continuous, it must remain so, and ψ also.

4. The Addition Equations I. and II. can also be discussed as follows:

Take
$$\left.\begin{aligned} f(x)&\equiv i\phi(x)+\psi(x)\\ g(x)&\equiv -i\phi(x)+\psi(x)\end{aligned}\right\} \quad \text{- - -} \quad (18)$$
$$\begin{aligned}\therefore\ f(x+y)&=i(\phi x\psi y+\psi x\phi y)+\psi x\psi y-\phi x\phi y\\ &=(\psi x+i\phi x)\,(\psi y+i\phi y)\\ &=f(x)f(y) \quad \text{- - - - - - - -} \quad (19)\end{aligned}$$
Similarly $g(x+y)=g(x)\,.\,g(y)$ - - - - - - - (20)

Hence $f(x)\equiv a^x$ and $g(x)\equiv b^x$, where a and b are arbitary constants.

Hence

$$\left.\begin{aligned}\phi(x) &\equiv \frac{1}{2i}(a^x - b^x)\\ \psi(x) &\equiv \frac{1}{2}(a^x + b^x)\end{aligned}\right\} \quad - \quad - \quad - (21)$$

There is no restriction as to the values of a and b, as we can verify by substituting in I. and II.

But if ϕ and ψ are to be *real* functions of x, since by (21) $a^x = \psi + \phi i$, $b^x = \psi - \phi i$, it follows that a and b must be complex quantities having the same modulus, say A, and equal and opposite amplitudes, $\pm$B.

Hence $\qquad \psi(x) = \mathrm{A}^x \cos \mathrm{B}x, \ \ \phi(x) = \mathrm{A}^x \sin \mathrm{B}x \quad - \quad - \quad - (22)$

Although this result agrees with the general result obtained by the previous method, it is to be noted that we have not proved exactly the same thing. Here we have identified $\mathrm{A}^x\cos\mathrm{B}x$ and $\mathrm{A}^x\sin\mathrm{B}x$ *as defined analytically* with the functions ϕ and ψ. In the previous method the analytical expressions for sine and cosine were not assumed.

On the Number and Nature of the Solutions of the Apollonian Contact Problem.

By R. F. MUIRHEAD, M.A., B.Sc.

The problem of describing a circle to touch three circles, including the nine special cases when one or more of the radii of the given circles are zero or infinite, was solved by Apollonius of Perga in a work which was lost, but of which Pappus has given some account in his Mathematical Collections. Towards the end of the 16th century the problem was again taken up and solved by F. Vieta, and since that time it has formed the subject of investigations by many mathematicians, from many different points of view.

The question as to the number and nature of the solutions in the different cases has not been neglected, and many authors have given more or less extensive tabular summaries of the number of solutions in various cases. One of the most extensive of these tables is that given by L. Gaultier de Tours in his paper in the *Journal de l'Ecole Polyt.*, *Cahier* 16, published in 1813. There he considers the problem of finding a circle to satisfy three given conditions of various sorts. He distinguishes 33 varieties of the problem, as well as 74 varieties of the corresponding problems with reference to the sphere, and tabulates the number of solutions. This table includes the ten cases of the Apollonian problem.

But, as this author points out, the numbers given are only true for certain relations between the data, and it is a matter of difficulty in some cases even to choose the data so that all the theoretical solutions shall be really possible. He makes no attempt to classify the cases according to the relative positions of the given circles, etc., and so far as I know, the only author who has supplied this omission is O. Stoll, who has treated the question pretty fully in the *Math. Ann.*, Vol. VI., p. 613, but from the analytical point of view. This paper will be referred to later.

§ 1.

Consider first the problem of finding the number and nature of the solutions of the Apollonian Problem in the special case when one of the three given circles is a point, and when the contacts with the circles are to be similar, *i.e.*, both external or both internal.

Let there be two mutually exclusive circles A and B (Fig. 44) and let their external common tangents t, t' be drawn. The whole plane is divided into portions X, X, Y, Y, Z, Z, A, B. Take now a circle having a mutually external contact with A and B, and imagine it to pass through all possible positions successively, begining with the tangent line t (which is a circle of infinite radius), and ending with the line t': the whole of the spaces X, X will have been swept over *once*, and the whole of the spaces Y, Y *twice* over; that is, each point in Y lies on the variable circle in two distinct positions.

Again, the circle touching A and B *and containing both*, sweeps out X, X once and Z, Z twice over. Note that the points within A and B are not swept over at all, and the points on the circumference are traversed *once*. Or we may say that as the point moves from *without* A inwards, the two contact circles which pass through it become *coincident* when it reaches the circumference, and *imaginary* when it has passed the circumference.

When the point lies on a common tangent, that tangent itself forms one of the contact circles in question, and may be reckoned as having either *external* or *internal* contact with both A and B.

We can now determine for each given point of the plane the number and nature of the solutions of the problem: *To draw a circle through this given point so as to touch the two given circles A and B in the same manner, i.e., both externally or both internally.*

We shall adopt a notation for distinguishing the different kinds

[redacted]

§ 2.

To treat the corresponding problem when the contacts are *dissimilar*, we must take the internal instead of the external common tangents. By *dissimilar* contacts we mean those in which one is external and the other internal, *i.e.*, one is denoted by α and the other by β or γ. The results are shown for the three cases in Figs. 47, 48, 49.

§ 3.

We now combine the results of the two previous §§, and thereby determine the number of solutions of the general problem to find a circle to touch two given circles and a given point, there being no restriction as to the nature of the contact. The resulting diagrams are given in Figs. 50, 51, 52.

Note that when the two given circles cut one another, there are *two* solutions wherever the given point may be ; on the other hand, when A and B do not cut each other, there are *four* solutions or *none* according to the position of the given point.

§ 4.

We proceed to extend the above method to the general Contact Problem of Apollonius, in which there are three circles given, and it is required to find one which will touch all three.

In doing so we use the artifice of "*parallel translation of circles.*"

Take first the case in which A, B, and C are the three given circles in decreasing order of size, and where A and B do not intersect, and let us seek the solutions in which the required circle has *similar contact* with A and with B.

Let A_0 and A_1 be circles concentric with A, having radii equal respectively to the difference, and the sum, of the radii of A and C. Similarly for B_0 and B_1 ; and let C_0 be the centre of the circle C. To any circle which touches A, B, and C externally, and whose contact is therefore of the nature $\alpha\alpha\alpha$, there is a corresponding parallel circle passing through C_0 and touching A_0 and B_0 externally. Conversely, for every circle touching A_0 and B_0 externally and passing through C_0, there is a parallel circle of smaller radius touching

A, B, and C externally. And for every circle touching A_1 and B_1 externally and passing through C_0, there is a circle touching A and B externally and C internally, which may be denoted by $\alpha\alpha\gamma$.

The general rule for the nature of the contact of the required circle with C is that if the radius of the circle passing through C_0 has to be *increased* to get the concentric circle touching C, that circle will contain C. If, however, the radius has to be *decreased* (algebraically) the contact circle will *exclude* or *be contained by* C, according as the radius of the circle passing through C_0 is *greater* or *less* than that of C.

The results are shown in Fig. 53, and the corresponding results for circles which have *dissimilar* contact with A and B are shown in Fig. 54. In these figures the dotted lines represent A_0 B_0 A_1 B_1 and their common tangents, and they are the boundaries of the various portions of the plane considered. The symbol $2\gamma\gamma\alpha + 2\gamma\gamma\gamma$, for example, indicates that if the centre of C is in the portion bounded by dotted lines, and thus marked, then there are four circles which will touch A, B, and C, the contacts with A and B being similar—two of these containing A and B and excluding C, the remaining two containing all three given circles.

We may note that the ring-shaped spaces between the two circles have two solutions in either of the figures 53, 54, while the external space has four, and the internal spaces none, so that on the whole there will be in this variety of the figure, 8, 4, or 0 [redacted]

This has been pointed out by O. Stoll in the paper above referred to, as well as the importance of the *common tangents* in the geometrical interpretation of his results.

Figures 53 and 54 in combination show the number and nature of solutions of the Apollonian Problem for the case in question: but it must be noted that these figures would vary not only in shape but in topologic nature for different relative positions of the circle. Thus the circle A_1 and B_1 might intersect, or the exterior centre of similitude of A_0 and B_0 might lie without B_1, etc.

The number of possible different cases of these figures is very great. And if we drop the restrictions that A does not cut B and that C is the least of the three circles, the number of different diagrams to which this method would give rise is enormous.

We may however consider the question from a different point of view, from which it will appear that in most of the different cases the number and nature of the solutions can be told almost at a glance by certain simple considerations, leaving a residue to which the foregoing method is more appropriate.

§ 5.

The topologically different relative positions of three circles in a plane may be classified as follows into 14 cases, *special* cases being for the present excluded.

Division I. Circles not intersecting.

- α. All three circles mutually exclusive.
- β. One contains the other two, which are mutually exclusive.
- γ. One contains another and excludes the third.
- δ. One „ „ which contains the third.

Division II. One pair only intersect one another.

- α. The third circle excluded by both of these.
- β. „ „ „ containing „ „ „
- γ. „ „ „ contained by „ „ „
- δ. „ „ „ contained by one of these and not by the other.

Division III. One pair only do *not* intersect.

α. This pair mutually exclusive.

β. One of the pair contains the other.

Division IV. Each circle intersects both the others.

α. No circle contains a crossing point of the other two. (No area common to all.)

β. One only contains two crossing points of the others. (The area common to all is two-sided.)

γ. One only does not contain two crossing points of the others. (The common area four-sided.)

δ. Each contains one crossing point of the other two. (The common area three-sided.)

[This classification agrees with that obtained by Stoll from analytical considerations, except that he does not distinguish between the four cases of Division IV., while he distinguishes two cases of III. β, according to the relative size of the radii. He gives a table showing the *number* of solutions for each of his cases.]

Now the shapes of the different portions of plane space in these figures are of seven different kinds, which can contain from 0 to 8 contact circles, thus: according as a space (which may or may not [redacted]

TABLE I

Ref. No.	Figure.	No. of Spaces Bounded by: 1 circle.	2 circles.	2 arcs.	3 arcs.	4 arcs.	1 circle & 2 arcs.	3 circles.	No. of Spaces.	No. of Contact Circles.	Nature of Contact Circles.
I. α	55	3	0	0	0	0	0	1	4	8	Various cases, see § 4.
„ β	56	3	0	0	0	0	0	1	4	8	$2\beta\gamma\alpha+2\beta\alpha\gamma$ $+2\beta\alpha\alpha+2\beta\gamma\gamma$
„ γ	57	2	2	0	0	0	0	0	4	0	
„ δ	58	2	2	0	0	0	0	0	4	0	
II. α	59	1	0	3	0	0	1	0	5	4	$2\alpha\alpha\alpha$ or $2\gamma\gamma\gamma$ or $\alpha\alpha\alpha+\gamma\gamma\gamma$ also $2\alpha\alpha\gamma$ or $2\gamma\gamma\alpha$ or $\alpha\alpha\gamma+\gamma\gamma\alpha$
„ β	60	1	0	3	0	0	1	0	5	4	$2\beta\alpha\alpha+2\beta\gamma\gamma$
„ γ	61	1	0	3	0	0	1	0	5	4	$2\beta\beta\alpha+2\beta\beta\gamma$
„ δ	62	1	0	3	0	0	1	0	5	4	$2\alpha\beta\alpha+2\alpha\beta\gamma$
III. α	63	0	0	4	0	2	0	0	6	4	$2\alpha\beta\alpha$ $+(2\alpha\alpha\alpha$ or $2\gamma\gamma\gamma$ or $\alpha\alpha\alpha+\gamma\gamma\gamma)$
„ β	64	0	0	4	0	2	0	0	6	4	$2\beta\alpha\beta+2\beta\alpha\alpha$
IV. α	65	0	0	3	2	3	0	0	8	8	$(\alpha\alpha\alpha+\gamma\gamma\gamma$ or $2\alpha\alpha\alpha)$ $+2\alpha\beta\beta+2\beta\alpha\beta+2\beta\beta\alpha$
„ β	66	0	0	3	2	3	0	0	8	8	$(2\alpha\alpha\alpha$ or $2\gamma\gamma\gamma$ or $\alpha\alpha\alpha+\gamma\gamma\gamma)$ $+2\beta\beta\alpha+2\alpha\beta\beta+2\alpha\beta\alpha$
„ γ	67	0	0	3	2	3	0	0	8	8	$2\beta\beta\beta+2\alpha\alpha\beta$ $+2\beta\alpha\alpha+2\beta\alpha\beta$
„ δ	68	0	0	0	8	0	0	0	8	8	$\beta\alpha\beta+\alpha\beta\alpha+\beta\beta\alpha+\alpha\alpha\beta$ $+\beta\beta\beta+\alpha\beta\beta+\beta\alpha\alpha+(\gamma\gamma\gamma$ or $\alpha\alpha\alpha)$

§ 6.

Class A. Two of the given circles touch each other. There are 16 cases which may be classified as A1., A2, or A3., according as the number of points of intersection is 0, 2, or 4. We shall denote by A1. α, β, γ, δ, ϵ, ζ, η, A2. α, β, γ, δ, ϵ, A3. α, β, γ, δ the sixteen cases shown in Figures 69 to 85, respectively.

In general such cases are limiting ones, separating cases of intersecting circles from those which have fewer intersections, or none. Thus A1. α separates cases I. α and II. α of § 5, and has 6 solutions (the mean of 8 and 4). Two of these are circles which touch the two given touching circles at their point of contact, each representing two solutions in the general case I. α which coincide for this limiting case, and disappear in II. α. The other four are the same as the solutions of II. α.

TABLE II.

Reference No.	Figure.	Cases Separated.	No. of Solutions.
A1. α	69	I. α and II. α	6
β	70	I. β and II. β	6
γ	71	I. γ and II δ	2
δ	72	I. γ and II, α	2
ϵ	73	I. δ and II. β	2

[redacted]

Table II. gives a summary of the corresponding results for this and the other cases of Class A.

One way of tracing the change through a special case from one general case to another, is to treat it from the point of view of § 1, taking for A and B the circles which do not touch.

The nature of the solutions in each case can be worked out on the same principles as have been employed above. Or we might deduce them from Table I. For example, A1β separates I. β from II. β. The solutions $2\beta\alpha\alpha + 2\beta\gamma\gamma$ are common to both, and therefore belong also to A1β; while the solutions $2\beta\gamma\alpha + 2\beta\alpha\gamma$, which exist in I. β but not in II. β, reduce to $\beta\gamma\alpha + \beta\alpha\gamma$ in A1β.

Class B. The three given circles concurrent in *one* point. Here there are three cases as shown in Figures 86, 87, 88. Each case has 5 solutions, one being a point circle coincident with the point of concurrence, which is the degenerate representative of four circles in the more general case. These three cases are respectively intermediate between IV. α and IV. δ, IV β and IV. γ, and IV. β and IV. δ.

Consider now the *doubly* special cases.

§ 7.

Table III.

Reference No.	Figure.	Cases Separated.	No. of Solutions.
C1. α	89	A1α and A2α	5
β	90	{ A1β and A2β or A1η and A2ϵ	5
γ	91	A1η and A2γ	5
δ	92	{ A1γ and A2α or A1δ and A2γ	3
C2. α	93	A2α and A3γ	5
β	94	{ A2γ and A3γ or A2β and A3β	5
γ	95	A2ϵ and A3γ	5

Class C. One given circle touches both of the others. 1. The other two intersect. 2. They do not.

The various cases are given in Table III.

Note that, as before, the number of solutions is the mean of the numbers corresponding to the two more general cases separated. In each case the circle which touches each of the other circles is itself one of the solutions and really represents *two coincident solutions*, as also do the other solutions where the circle sought has common contact with two of the given circles. Thus in Figure 96 which indicates the solutions of case C1. *a*, the circles 3, 4 and B each represent two solutions of the more general case I. *a*.

Class D. Combining the conditions of classes A and B. Two cases, shown in Figures 97 and 98. In either case the number of solutions is 3, including the point circle coinciding with the point of concurrence, which is the degenerate representative of 6 circles.

Class E. The three given circles concurrent in *two* different points. One case, Figure 99, where the only two solutions are the two point-circles at the intersections, each of which represents four circles of the general case IV.

Class F. *Trebly* Special Case. The three given circles have a common point of contact. Figures 100, 101. Here the number of

[redacted]

TABLE IV.

	Given Circles.	Given Lines.	Given Points.	Greatest Possible No. of Solutions.
(1)	2	1	0	8
(2)	1	2	0	8
(3)	0	3	0	4
(4)	2	0	1	4
(5)	1	1	1	4
(6)	0	2	1	2
(7)	1	0	2	2
(8)	0	1	2	2
(9)	0	0	3	1

Thus the maximum number of solutions is 2^{3-n} where n is the number of given points, except in Case (3) where we may say that four of the circles have gone off to infinity, and in Case (6) where in general the lines intersect one another and not the point, so that it belongs to Class II. which has only *half* the maximum number of solutions of the general case.

All these 9 cases are susceptible of discussion by the method of §§ 1, 2, 3, 4.

If in Fig 44 of § 1 we imagine the greater circle to become infinite, then the external common tangents become coincident with a line parallel to A touching B on the side remote from A. The regions X X disappear, the closed space Z vanishes, and the open space Z goes to infinity, and the whole of the space between A and B belongs to Y Y, so that if C lies in this region, the solutions are denoted by $2\alpha\alpha$, while if C lies elsewhere there is no solution.

Taking the corresponding case of § 2, viz., Fig. 47, we see that a similar result comes out, the solutions for the space between A and B being denoted by $2\alpha\gamma$, *i.e.*, there are two contact circles *containing* B. Thus in Case (5) when the line and circle do not intersect, there are two contact circles which exclude B, and two which contain B.

Note that in treating a line as a circle of infinite radius we may arbitrarily choose *either* side of the line to be the *interior*: but in order to apply the previous results, we must consistently keep to the choice made.

We might arrive at the results just found more directly, by observing that the variable circle which touches A and B and excludes B passes over every point of the region between the two *twice over*, as also does the variable contact circle which contains B.

Taking cases (1) to (9) *seriatim*:—

In (1) we note that the various cases are as a rule special cases separating one case of Table I. from another, though some of them are special forms of a single case of Table I. The results are shown in Table V.

TABLE V.

Figure.	Cases of Table I. Separated.	No. of Solutions.	Nature of Solutions.
102	I. α, β	8	$2\alpha\gamma + 2\gamma\alpha + 2\gamma\gamma + 2\alpha\alpha$

[redacted]

Case (2). Here I. can only exist in a sub-case, *i.e.*, when the lines are parallel. There are 4 solutions if we have II. (or III.). *i.e.*, if the lines are not both cut by the given circle, and 8 solutions if we have IV., *i.e.*, if both lines cut the given circle. The nature of the contacts is obvious in the several varieties of this case.

Case (3). This belongs in general to IV., but four of the circles have gone off to infinity, leaving only four solutions, the in- and escribed circles of the triangle.

Case (4). This has already been treated in §§ 1, 2, 3.

Case (5) When the line and circle do not intersect, we have as at the beginning of this §, four solutions $(2\alpha + 2\gamma)$ or none; and when the line and circle do intersect, two solutions (2α or 2β according as the point is without or within the given circle).

Case (7). We might treat this by §§ 1, 2, taking B to be a point-circle, in which case the interior region Z vanishes, as in Fig. 114. Thus if C lies in 7, there are two solutions (2α); if in Z, 2γ; if in X, $\alpha + \gamma$. In other words there are two exterior contact circles if A is met by BC produced but not by BC, two interior contacts if BC cuts A twice, and one of each if neither BC nor BC produced cuts A, and none if the circumference of A separates B and C.

The same results could be got by considering the variable circle passing through B and C.

Cases (6), (8), (9) are all obvious.

Each problem of § 8 might be considered under the special conditions of Classes A to F and thus give rise to yet more special cases; and there would be no difficulty in applying the same principles as we have used above to determine the number and nature of their solutions.

There are also many other kinds of special cases, *e.g.*, when intersecting lines become parallel, etc.

On the other hand the above methods may be applied to the corresponding Contact Problem for Spheres instead of Circles. In place of the common tangent lines of two circles, of §§ 1, 2 we should have to consider the *common tangent planes of three spheres.*

Seventh Meeting, June 12th, 1896.

Dr Peddie, President, in the Chair.

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Erste Abtheilung 1668--1699 (1894).
Zweite Abtheilung 1700--1728 (1896).

A Review: with special reference to the Rise of the Infinitesimal Calculus and the Newton-Leibnitz Controversy.

By Professor George A. Gibson.

The gigantic task which Mr Cantor has undertaken in writing a history of mathematics down to the year 1759 is approaching its accomplishment, the first two of the three parts forming the third and concluding volume being now published. How great is the debt of gratitude that the mathematical public owe to him for the erudition and thoroughness he has brought to bear on the work, only those can guess who have attempted to follow out some line of historical investigation. His history is totally different [redacted]

examination of the various chapters in these two parts would demand more space than our *Proceedings* can afford, even if I were competent to make it, and I propose therefore to deal mainly with the rise of the Differential and Integral Calculus and the controversy between Newton and Leibnitz in regard to the invention. It would seem as if every document of importance, so far as these are ever likely to be found, were now before the world; and the judgment of the mathematical public respecting the controversy might be expected to be formed in as definite a way as the nature of the dispute admits.

It is now, I think, generally recognised and in any case it is emphasised by Mr Cantor that the second half of the seventeenth century was ready for the instrument that we denote by the Infinitesimal Calculus and that the writings of men like Cavalieri, Fermat, Pascal, Wallis contain many processes and results that would now be classed as characteristic of that Calculus. The question therefore that naturally arises is, what was the gain to mathematics from the work of Newton and Leibnitz in the field of the infinitesimal calculus? As regards the controversy between the two, there is the further question, did the one derive from the other ideas, suggestions, or results that were of importance for the development of his system?

In the closing paragraphs of his second volume Mr Cantor has stated that the half century from 1615 to 1668 may be considered to be that of the discovery of the Infinitesimal Calculus; first, problems of the Integral Calculus were treated, next problems of the Differential Calculus, then problems in which both appeared, but the intrinsic connection between the two branches was only dimly perceived. The task that devolved upon the mathematicians who succeeded these pioneers was to make explicit this connection and to provide an instrument of research in the shape of an appropriate notation and algorithm. On p. 125 of Vol. III. Mr Cantor repeats this summary, and lays stress on the discovery of a notation as the condition for further advance, while apparently considering as of less importance the appreciation of the relation between the two branches. At any rate, throughout the analysis and discussion of the writings of Newton and Leibnitz it is the presence or absence of a notation and an algorithm, "of a grammar of the language" that he most frequently insists upon.

The predecessors of Newton and Leibnitz had given many examples of infinitesimal operations; those of the differential calculus appeared for example in the drawing of tangents, and in the investigation of maxima and minima; those of the integral calculus in quadratures, &c.; while the inverse method of tangents was closely related to differential equations. In the beginning of our study of the Calculus now-a-days we learn first the simpler operations of differentiation and we usually (unfortunately) advance pretty far in the differential calculus before broaching the integral; but when we do come to integration we treat it as the inverse of differentiation. Now this inverse character of differentiation and integration had to be seen before there could be much of an integral calculus; in other words, only thus could the essential identity of problems in the direct and inverse method of tangents and in quadratures be perceived. I fully agree with Mr Zeuthen *(Bulletin de l'Acad. Roy. de Danemark*, 1895, No. 2) that the recognition of this inverse character is a fundamental condition for the rise of the Calculus, as a system; and Mr Cantor, while not forgetful of it, has perhaps laid too much stress on the invention of notations and algorithms.

Again in the development of a calculus that was to go much beyond the earlier results there was one essential instrument, viz., the binomial theorem for any index. Without this theorem and the explicit use of fractional and negative indices the algorithm of the new calculus would have been very imperfect and advance very [redacted]

of series as forming for Newton an indispensable adjunct to his calculus. Integrations in finite form are important, but without infinite series the structure of the Calculus, whether viewed theoretically or practically, would be stripped of some of its most valuable features. In considering then the contributions of Newton and Leibnitz to the development of the Calculus we must look not merely at the formal side—the construction of a notation and algorithm, but also at the material—the discovery of the results that make it valuable.

Mr Cantor in view of the miserable controversy about the invention of the Calculus and the influence on it of the personal circumstances of the two great rivals narrates the story of their life in unusual detail.* In this country we are perhaps more familiar with the personal characteristics of Newton than with those of Leibnitz. † As regards the latter, it is perhaps too often forgotten that the study of mathematics was not the principal pursuit of his life and that at what may be called critical periods he was often overwhelmed with duties of a totally different kind. Nor did he owe much of his mathematical skill to his University training; the account of one of his teachers, Erhard Weigel (pp. 36–7) is a startling revelation of the state of mathematical instruction in the German Universities of the period. As Leibnitz himself says, he was when he came to Paris in 1672 self-taught in mathematics and had only a superficial acquaintance with the higher branches. His introduction in 1673 to Huygens was the turning-point in his mathematical career and he always gratefully recognised how much he owed him for direction and stimulus. I feel obliged to add that I think Leibnitz possessed in a pretty high degree faults that are too often inherent in the self-made man; his self-consciousness is excessive and the self-satisfied tone in which he speaks of his achievements, though at times amusing, is frequently offensive. In this respect he presents a striking contrast to Newton.

The writings of Newton that are of most importance for our purposes are (1) *De Analysi per aequationes numero terminorum infinitas*, sent to Collins in 1669 but first published in 1711; (2) *De Quadratura curvarum*, published in 1704; (3) *Methodus*

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* The statement on p. 65 that the binomial theorem was engraved on Newton's tombstone seems to be a mistake.

† According to Mr Cantor we should write Leibniz.

Fluxionum et Serierum infinitarum, published in 1736. Besides these are three letters, one to Collins in 1672 that may be called "the tangent letter" and two to Oldenburg (to be communicated to Leibnitz) the first dated 13th June 1676 and the second 24th October of the same year. (1) was printed in 1711 in the form in which it was sent to Collins but it was probably drawn up in 1665–6. Whether (2) and (3) underwent revision before publication can not be definitely decided, but it seems certain that the substance of (2) dates back to about 1666 and in any case not later than to 1676 while (3) was probably written, nearly as we have it, in 1671. The reasons urged on p. 165 in favour of a revision of (3) are not, I think, well founded (see Mr Zeuthen l.c.), though Mr Cantor is on stronger ground when he argues (p. 171) for an addition later than 1673.

To what extent Collins made known the contents of the *De Analysi*, it seems difficult to determine; the letters from Collins printed in the *Commercium Epistolicum* contain several of the series and intimate that Newton has a general method, applicable to quadratures, rectifications, drawings of tangents, &c., but they give no clue to what that method is. James Gregory in replying (15th February 1671) to one of Collins's letters says he thinks he knows Newton's general method to some extent and communicates certain series of his own, among them that for $\tan^{-1}x$; Collins (21st February 1671) writes that Gregory "has very recently fallen upon the same [redacted]

also considered.* Newton further remarks that, by assuming any equation whatever between z and x, y may be found in the same way; *e.g.*, if $z = \surd(a^2 + x^2)$, then $y = \dfrac{x}{\sqrt{a^2 + x^2}}$.

B. If the expression for y is not a simple sum of powers of x, it is expanded in series and to each term the general rule is applied. To obtain the series, the binomial theorem (or rather divisions and root extractions) is applied if y is an explicit function of x; but when y is an implicit function it is expanded, in ascending or descending powers of x, in the now well-known manner, and the question of convergence of series is not passed over.

C. Newton's principle of "moments" is stated and applied to finding the length of a circular arc; this section seems to me to be one that the first readers of the tract would find very difficult.

D. The number and the importance of the series obtained are very striking; besides the binomial theorem, there are the series for $\sin x$, $\cos x$, $\sin^{-1}x$, e^x, $\log(1+x)$, in fact all the important elementary series. The expansion of y when given as an implicit function of x and the reversion of series are very important methods which are sketched in a most instructive manner.

In the tract the general method of measuring "the quantity of curves" is professedly "briefly explained rather than accurately demonstrated," but the closing sections return to the proofs. It would be too much to say that they meet modern demands, but they are the work of one thoroughly imbued with the spirit of accuracy characteristic of the ancient geometers.

It seems to me beyond all question that Newton has here made a new departure; the relation, as we would say, of integration to differentiation is clearly expressed and the method is general because it embraces problems which were formerly solved each by its own special method. It is, as Mr Cantor says, quite true that there is no trace of the differentiation of products and quotients, nor is there any special notation; but there is a working system such as had not yet appeared, shedding a new light on the old problems and capable of results of the most important kind. In fact, what we understand by the "Calculus" has now been discovered, and the whole

* Mr Cantor's phrase (p. 151) "perhaps also negative" is not justifiable.

style of the arrangement and composition proves that Newton has only given a sample of the stores in his possession.

Next I should say something of the other two treatises, but as regards the relation of Newton to Leibnitz these do not occupy the same position as the first one. It is important however to notice that they completely bear out all the claims Newton puts forward in the tangent letter and in the letters to Oldenburg. The *De Quadratura* is as much a treatise on integration as on what we call quadrature, and is a very thorough piece of work, containing as it does standard forms and groups of integrals which prove how fully Newton had realised the importance of his method and the necessity of having the standard results at command. I think Mr Cantor has dwelt too much on the slips Newton occasionally made; they bulk too largely in the History. Mr Cantor's admiration for Newton's work is sincere and I can hardly think he is quite aware of the impression his frequent reference to the slip (for I think it is a slip) regarding the relation of fluxions and increments will leave on readers of his book. Again his criticism (p. 165) of Newton's procedure in making the fluxional equation homogeneous seems to me unreasonable, nor I think does Mr Zeuthen quite meet the objections. It seems to me from Newton's repeated observations in the *Methodus fluxionum*, that a fluxional equation is for him simply meaningless unless it is homogeneous in the fluxions, exactly as Leibnitz argues for homogeneity in differentials. There is an apparent arbitrariness in introducing a new fluxion, but the real meaning of the process

fluxion of an expression in the *Methodus Fluxionum* (Opuscula I., p. 55). In both cases he speaks of "any arithmetical progression whatever"; the meaning of this phrase is not quite clear (but see Op. I., 61), though the correct progression is used in the example. The peculiar phrase noted, though making the rule less clear and even inaccurate for the particular example, may have been used in view of the rest of the letter in which Newton says that the rule is a mere corollary of a general method, and is, unlike Hudde's method for maxima and minima, applicable even when the equation contains surds, while extending also to more difficult problems in curvature, areas, &c. From this letter I do not think any one could learn more of the method of tangents than he might from Sluse or Hudde; there is only this other information that Newton claimed to be able to do more than these two. It is a bold statement that "in this letter the method of Fluxions was sufficiently described to any intelligent person."

Reading the two letters to Oldenburg and the tangent letter in the light cast on them by the *De Quadratura* and the *Methodus Fluxionum* one can hardly doubt that by 1676 Newton was in possession of his new analytical methods almost as he left them. With him notation and algorithm are of very secondary importance, and it was only after much fluctuation that he adopted the notation he actually published; the superiority of the Leibnitzian calculus in this respect, though denied by Newton and his immediate successors, must be held as proved by the subsequent development of the higher mathematics. On the other hand that development owed much to Newton's work in the field of infinite series where he was an acknowledged master, Leibnitz himself always admitting Newton's merits here even when embittered by the priority controversy. It is not easy to determine how far Newton himself distinguished between the method of fluxions and the method of infinite series, and it was always a sore point with Leibnitz that the English adherents of Newton turned the priority dispute on matters of series in which he granted Newton's priority and pre-eminence. In their own day and for long after, a distinction was recognised, though infinite series formed an important element in many applications of the calculus and were really essential to its progress. The restriction of the calculus to operations which were effected on, or resulted in, expressions in finite form is very arbitrary; but that

some such restriction was made is implied in the manner in which writers of that time spoke of the subject. Newton, however, after laying down in clear terms the fundamental principle of the formation of fluxions and of returning in the simpler cases from the fluxions to the fluents proceeds to apply that principle by a systematic use of infinite series, but is also on the watch for closed expressions as particular cases. The integration, as we would now say, of the binomial differential (stated in the 2nd letter to Oldenburg and proved in the *De Quadratura)* is a remarkable instance.

Whether Newton fully realised the importance of his work on fluxions and series before the brilliant results of Leibnitz and the two Bernoullis showed the importance of the method, can not be definitely settled ; that he valued it highly is clear from the letters to Collins and Oldenburg, but it would almost seem as if the enormous labour he bestowed on series had produced a sort of reaction. This feeling comes out in the two letters to Oldenburg and still more strongly in his communications with Wallis, where he spoke of himself, Wallis says, as *harum rerum pertaesus*, leaving to others who might have the leisure and the will an open field in which to exercise themselves. (Letter to Leibnitz, January 16th, 1699.) His aversion to controversy prevented him from publishing an account of his methods, as he had intended to do, in 1671 ; but no one, I suppose, will question that his method of fluxions is essentially the same as the differential method of Leibnitz and that [redacted]

prominent place in his conceptions and, as he wrote at a later date (1693), only repeating a remark found in a manuscript dated 26th March 1676, "one part of the secret of analysis consists in the characteristic, that is to say in the art of employing well the symbols we use." His earliest work was in the theory of combinations and a little later he is found working at numerical series. Early in 1673 he visited London but did not on this occasion, it would seem, make the acquaintance of Collins. When in London he met the mathematician Pell and in the course of conversation mentioned a method he had of treating series, which he called the method of "generating differences"; but Pell pointed out that it was already known. On getting the book by Mouton which Pell referred to, Leibnitz found the correctness of Pell's statement. In order however to clear himself from any possible charge of plagiarism he addressed a formal letter to Oldenburg explaining how he had come upon his own method and pointing out that he went further than Mouton. This incident was afterwards adduced in proof of Leibnitz's plagiarising propensities, though I hardly think with reason. At a later period Leibnitz always declared that his studies in the method of generating differences gave him a clue to his differential method.

In London he got Mercator's Logarithmotechnia (published in 1668) and the study of this work and of other works recommended to him by Huygens led him to what he calls a general method of transformation by means of which any given figure is "transmuted" into an equivalent one involving an equation of no higher degree at most than the third. I cannot say I think he has proved what he claims for this method of "transmutation" (as he calls it) but at any rate it enabled him to find series for the areas of sectors of central conics. One of these series is the same as Gregory's series, and here again a charge of plagiarism has been made. But Leibnitz's mode of proof is that of Mercator rather than that of Newton and a letter of Huygens shows that Leibnitz sent him the series (for $\pi/4$) before he had any means, so far as we know, of seeing Gregory's series. Such coincidences are very unfortunate for Leibnitz in view of the later dispute, but Mr Cantor's discussion proves, it seems to me, Leibnitz's honesty in the two cases.

The most important part of Leibnitz's mathematical studies during his residence in Paris from 1673 to 1676 consists in his

researches on the borderlands of the calculus. Since the most convincing proofs of Leibnitz's independence are the contents of his notebooks of this period, it is a painful but a necessary duty to state that a falsification of date has been detected in one of his MSS., 1675 being altered to 1673. Mr Cantor, with his usual fairness, mentions this (p. 176) and assumes that it is the act of Leibnitz himself. At the same time I think the conclusions he draws are sound;—(1) that the detection of one falsification would induce a careful scrutiny of the other dates and as no further tampering has been noticed we may conclude that the other dates are genuine; and (2) that the date of 1675 is definitely established.

Assuming then the genuineness of these MSS., I think we may hold it as proved that before he had any information regarding Newton's methods and before his second visit to London in 1676 Leibnitz was already on the tracks of his general method. It may be noted in passing that the procedure of the earlier analysts for drawing a tangent lay in finding the subtangent. The right-angled triangle formed by an infinitesimal arc and the differences of the abscissae and of the ordinates of the ends of the arc is called by Leibnitz "the characteristic triangle" and is fundamental with him; it is obviously similar to the triangle formed by tangent, subtangent and ordinate. Whether he got the idea of this triangle from his study of Pascal (as he always maintained) or from Barrow, does not much matter—except as another instance of his duplicity; [redacted]

tangents provided their differences are known" (MS. of date 26th June 1676). In a MS. of the following July the solution of De Beaune's problem is found.

It seems to me characteristic of these MSS. that the results are usually got by beginning with known problems, solved or unsolved.

Leibnitz then may fairly be said to have laid the foundations of his calculus; I do not think however he had gone very far in the actual development of it. Further, I do not think that anything in the MSS. referred to above could be due directly to any communications from Oldenburg or Tschirnhaus; I accept Mr Cantor's conclusions on this point.

We now come to Newton's first letter, sent off by Oldenburg on 26th July 1676. Along with this was sent a *resumé* of the tangent letter which did not contain the example but stated Newton's claims for his general method, in particular that it did not stick at surds. Newton's first letter contained the binomial theorem and the statement of the use of fractional and negative indices; the method of expanding implicit functions; several of the series of the *De Analysi*: and several examples of quadratures and rectifications by series.

I do not think Mr Cantor has given due weight to the help this letter might have given to Leibnitz; so far as I can judge his algorithm made little progress till he became familiar with the binomial theorem and was sure of his fractional and negative indices. His efforts to secure expressions in finite form, which Mr Cantor thinks characteristic of Leibnitz, could have but little success without such instruments. And further, Leibnitz's characteristic of working from given problems was provided with a splendid field to exercise in.

To this letter Leibnitz replies on the 27th of August. He first explains his method of transmutation—not a lucid exposition by any means. He then proceeds to give series which occur in Newton's letter and seems to claim that he had found them for himself. Towards the end he mentions that he has lately solved De Beaune's problem—which passed the skill of Descartes but which he had solved within an hour by the help of a certain analysis. He adds however that in this field he has not yet effected much, though he knows it to be of the greatest moment.

In the letter his symbols do not carry their signs with them, and indeed for several years he did not grasp the generality of

method which Newton in his reply shows him is to be gained by letting the symbols include the signs. But what are we to say about the series which he seems to claim as having also been discovered by himself and of which he gives no proof? If he did discover them for himself, I cannot but think he was enabled to do it by means of the theorems Newton had given (without proof, as Mr Cantor insists) in the letter to which Leibnitz was replying. If he did thus discover them, he gives a strong proof of his ability to profit by comparatively obscure suggestions and at the same time discloses an unfavourable feature of his character in not saying that such was the origin of them. As Newton referred to this matter in his next letter, in somewhat reproachful terms, it was incumbent on Leibnitz to put him right on his method of procedure.

However that point may be settled, Newton was evidently greatly pleased with Leibnitz's communication and drew up, under date 24th October 1676, a very long letter in reply, but *when* the reply reached Leibnitz has been subject of dispute. In the end of October he was in London for about a week and it is probable that on this visit he saw at Collins's the MS. of the *De Analysi* and made those extracts from it that are now among his documents in the Royal Library at Hanover. That he should have seen at Collins's a letter addressed to himself, under cover to Oldenburg, and not have taken possession of it seems quite incredible and until there is evidence to the contrary I think we must assume that he got his first acquaint- [redacted]

to reply on the day he received it. Unfortunately the word *hodie* on Leibnitz's own draft (a genuine *hodie*, not of later insertion) was omitted, through some unexplained cause, on the copy that was current in England, and Mr Cantor dwells at great length on the probable effects of the omission.

In his answer Leibnitz makes his first open use of his differential notation and shows by examples that he is in possession of a method of tangents that does not stick at surds. The actual calculations are however affected with blunders in differentiating the surds that can hardly be set down as slips of the pen, though the preliminary explanations of obtaining tangents by differences are quite clear; the underlying principle was indeed quite common property by this time and the really new application was to surd quantities. Newton had claimed that he had surds in his power and Leibnitz wanted to show that he also had a method that did not stick at surds. Mr Zeuthen has an ingenious theory that Leibnitz owed his solution to a generalisation of the solitary example that occurs in the *De Analysi*, namely that if $z = \sqrt{(a^2+x^2)}$, $y = \dfrac{x}{\sqrt{(a^2+x^2)}}$. If this be correct, then Leibnitz had a marvellous power of profiting by suggestions, but I hardly think that this theory is sufficient. It is perfectly clear in any case that Leibnitz had very little command over fractional indices; he still used the old notation for roots of quantities; and there can be little doubt, as Mr Cantor supposes, that he thought $d(\sqrt[n]{x}) = dx/n(\sqrt[n]{x})$ without seeing that this contradicts the rule he had previously given, namely, $d(x^z) = zx^{z-1}dx$. But the mistakes in differentiating, which must soon have been discovered and actually were so, hardly affect the principle of the method, which is that the differential dx must come outside the root. Now Leibnitz himself in a letter to de l'Hospital (of a much later date, it is true, 27th December 1694) expressly says in explaining the origin of his calculus, that "he saw that the differential quantities would necessarily come outside the fraction or vinculum"; and comparing this statement with C (above) I am disposed to believe that he had guessed, even before seeing the *De Analysi*, that his method would extend to surds. He had, as I believe, worked, up till this period, very little with the more complex surd expressions; but he may well have reached the fundamental principle, either by substitution as in the letter, or by direct use of the binomial theorem. It

is perhaps worth noting that Keill, in his letter to Sloane, insists on the help Leibnitz got from the binomial theorem for the extension of his results beyond those of Sluse and uses the same expression as is found in the letter to de l'Hospital, namely, "where we see that the differential quantity $\frac{m}{n}cbx^{c-1}dx$ always remains outside the vinculum"; I fancy it was this property that first struck Newton himself.

Leibnitz does not refer in this correspondence to his having seen the *De Analysi* when in London, and Mr Cantor very justly censures his silence, while trying at the same time to explain it. His continued silence cannot but rouse suspicion and though his future frequent appeals to Newton to publish his general method induce one to believe that he was not conscious of making any unfair use of the lights he had obtained from Newton, it is not altogether unreasonable that suspicions should so long have been entertained of him.

Newton sent no reply to this letter of Leibnitz, and henceforward they work apart. Up to this point Leibnitz has shown great capacity and has, I think, given ample proof that the seeds of his new calculus are germinating in his mind and that he has, independently of Newton, got upon the right track. But he is a mere tyro as compared with Newton, and has given very little of actual mathematical results for the rich fund that Newton supplied. How the [redacted]

calculi genus. The differential dx is not defined as an infinitesimal, but is arbitrary and such that dy/dx expresses the ratio of the ordinate of a curve to the subtangent. Rules are stated but not proved for the differentiation of constants, sums, products, quotients, powers, roots; he uses substitutions for complex expressions; and he mentions a test for maxima and minima. It is clear that the paper excited considerable attention, as in a work * published at London in 1685 by John Craig, a Scotchman resident at Cambridge, the method of Leibnitz is approvingly referred to and used. In 1686 Leibnitz has another paper in the *Acta*, entitled *De geometria recondita et analysi indivisibilium atque infinitorum*, and here the integral sign appears for the first time in print. It is perhaps worth noticing that he gives the equation $y = \sqrt{(2x - x^2)} + \int \frac{dx}{\sqrt{(2x - x^2)}}$ which occurs in the *De Analysi*, and adds "this equation perfectly expresses the relation between the ordinate y and the abscissa x, and from it all the properties of the cycloid may be proved; and in this way the analytical calculus is carried forward to those lines which have hitherto been excluded for no other reason than that they were believed unsuited for it." It is not unfair to conclude that he may have been indebted in the same way to other results of Newton for the illustration of his calculus; not that there is anything to complain of in this, but it does show that he is not so independent of Newton as he seems to make out.

He also introduces in this article his distinction between transcendent and algebraical quantities and points out that his calculus is also applicable to the former, the equation just cited being an illustration.

A historical summary that Leibnitz gives is noteworthy because of the language of hearty commendation in which he speaks of Newton and of what Newton has "excogitated"; he believes that if Newton would only publish what he has by him he would without doubt open up new avenues to the great advance of science.

In 1687 appeared the *Principia*. That Newton made no explicit use of fluxions in this work is, to my mind, sufficiently explained by the reasons usually advanced; but of more importance for present

* In this work, *Methodus figurarum . . . quadraturas determinandi*, the binomial theorem appeared in print for the first time.

purposes is the celebrated scholium in which Newton states that in letters which passed between him and Leibnitz ten years before, he had mentioned that he was in possession of a general method for tangents, &c., which did not stick at surds but had concealed it under an anagram, and that Leibnitz in reply had said he too had fallen upon a method of that kind and communicated it, the method of Leibnitz hardly differing from his own except in the form of words and notations. The plain meaning of this declaration in my opinion is, that at the time of writing the *Principia* Newton believed he had concealed the principle of his method by means of the anagram, and that Leibnitz had hit upon his in some other way. In the second edition of the *Principia* (1713) another difference in the methods is noted, namely, "the idea of the generation of quantities"; whether this addition was due to Cotes or Newton can not be decided; its importance can not be questioned. But in the third edition (1726), when Leibnitz was dead and the dispute for the time dying away, the scholium was completely altered; Leibnitz's name was omitted and the tangent letter to Collins took the place of the former correspondence.

Mr Cantor considers the scholium of the first two editions as admission by Newton of Leibnitz's independence, and I do not myself see in what other light it can be read; the contention of the *Recensio* does not seem to me to touch the point in dispute. Newton afterwards believed that Leibnitz had practically appropriated his calculus; but that later belief, whether well or ill founded, is not

[redacted]

the leading ideas of his fluxional method, and here for the first time the fluxional notation appears in print. On seeing these letters John Bernoulli concluded, as he wrote to Leibnitz, that Newton's method, as mentioned in the *Principia*, does not really differ from that of Leibnitz and suggests that Newton may have constructed his calculus after seeing that of Leibnitz. Leibnitz puts him right on the matter, but the letter shows that Bernoulli had put the natural interpretation on the scholium.

The first public charge against Leibnitz in reference to the calculus was made by Nicolas Fatio de Duillier in a tract *Lineae brevissimi descensus investigatio*, published in 1699 under the sanction of the Royal Society. Fatio asserts that Newton had discovered the calculus many years before Leibnitz; "nor will the silence of the more modest Newton or the arrogance of Leibnitz, who everywhere attributes the discovery of this calculus to himself, deceive any who have gone through those documents which I myself have turned over." Fatio probably refers to the Oldenburg letters, but where he saw them we do not know.

Leibnitz was naturally incensed at the accusation, especially as it seemed to bear the mark of the approval of the Royal Society, and he applied to Wallis to help him in the matter. (Wallis's correspondence with Leibnitz was always of the most friendly character, though he was neither *nostras* nor *apud nos.*) Wallis answered that he had seen but not read Fatio's book, and that he disapproved of the attack whether it proceeded from Fatio or any one else. The approval of the Royal Society was due to an oversight, as would be seen from the accompanying letter of Sloane, the Secretary. Leibnitz then publicly replies to Fatio's charges in the *Acta*, for May 1700, and in the course of his answer appeals to the *Principia* as proof that Newton himself recognised his independence.

That Newton should not have known of this attack, in which the Secretary of the Royal Society had to intervene, is all but incredible; the reasons for his silence can only be guessed at. In the letters to Wallis and in the preface to the *De Quadratura*, published in 1704, that is on occasions both before and after Fatio's impeachment of Leibnitz when the opportunity of referring to the new calculus which was being cultivated so assiduously on the Continent presented itself, he acts as if for him that calculus did not exist. In spite of the provocation he had received by Leibnitz's

article in the *Acta* for 1689, *Tentamen de motuum coelestium causis*, the tone of his letter to Leibnitz in October 1693 is extremely friendly. Whether he sympathised with Fatio in his attack or not, it is impossible definitely to say. Up till this time his conduct seems to me beyond reproach, and seeing that he set such great store by his method of fluxions as to secure his claims to the discovery by enshrining its leading principles in the anagrams of the Oldenburg letters as well as in the scholium of the *Principia*, it remains a mystery why he did not step forward after the publication of Leibnitz's reply to Fatio. The article of 1689, referred to above, may possibly have aroused his suspicions (though the letter of 1693 is rather against that hypothesis) and he may not have felt so sure of Leibnitz's straightforwardness as to give a favourable answer to his appeal, while his known dislike of controversy would restrain him from pushing his own claims. By his silence however he put himself, I think, in an unfavourable position for bringing home to Leibnitz at any later time a charge of unwarranted borrowing.

If Leibnitz felt hurt at Newton's silence, he took a very dishonourable form of revenge by referring as he did to Newton in his review in the *Acta* for January 1705 of the *De Quadratura*; however much aggrieved he may have been, and I think he had good grounds for complaint, at Newton's silence in the affair of Fatio, he only demeaned himself by speaking as he did of Newton. The general tone of the article is favourable enough, though through-

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In seeking to explain, though not to justify, the review, Mr Cantor says all in Leibnitz's favour that can be said and his plea is not without weight; Leibnitz wished to make Newton speak out by making him feel how keenly a man suffers under an unjust reproach. This at any rate is the best that can be said for him, whatever weight we may attach to it.

From this review, which Newton does not seem to have heard of at the time of its appearance, sprang the bitter controversy, though the immediate occasion was given by John Keill, Professor of Astronomy at Oxford, who in an article in the *Phil. Trans.* Vol. XXVI. issued in 1710) inserted a sentence to the effect that Newton was the first inventor of the arithmetic of fluxions, which Leibnitz had later, with a mere change of name and notations, published in the *Acta Eruditorum*.

Leibnitz as Fellow of the Society received his copy of the *Phil. Trans.* when at Berlin in 1711, and immediately wrote to Sloane demanding that Keill should withdraw his imputations. On the 22nd March 1711, at a meeting of the Royal Society, with Newton in the chair, a part of the letter was read. Newton was at first displeased with Keill's action, but at a meeting held a fortnight later Keill directed Newton's attention to the unfair account of the *De Quadratura* in the *Acta* for 1705, and Newton thereupon explained how the matter stood, giving the time of his own earliest researches. Keill was then requested to draw up a statement on the matters in dispute. At the sitting of 12th April Newton referred to his letters to Collins; and Keill, who was present, was again asked to investigate the matter and defend the President's rights. Finally on the 24th of May Keill's answer was read, and on the 31st Sloane was instructed to forward Keill's defence to Leibnitz.

Keill's letter is addressed to Sloane. He begins by saying that he does not wish to deny to Leibnitz anything that is justly his, and he did not mean that Leibnitz knew of Newton's name and notations for his calculus; but he undertakes to prove that Newton was the first inventor, and that in his letters to Oldenburg and through him to Leibnitz, he had given such clear indications that a man of ordinary sagacity could not fail to deduce Newton's method. The review in the *Acta* is alleged to be the reason for Keill's action; for if it were right for the Leipzigers to credit to Leibnitz the

discoveries made by others, the British (Keill was Scotch !) might fairly demand back what had been taken from Newton.

I think Keill's letter is a much abler impeachment of Leibnitz than Mr Cantor admits. That Newton is the first inventor, Mr Cantor himself allows; and it seems to me that Keill seizes with remarkable skill on the portions of the letters from which Leibnitz might profit. But what Keill refuses to see is that only a man far advanced on the way to a calculus could avail himself of Newton's communications as Leibnitz might have done; unless Leibnitz had previously discovered the fundamental features of the calculus, the information given by Newton would not have borne the fruit Keill supposed it to do.

How far Keill's ignorance of the fact that Newton's second letter was answered immediately after its receipt is to be held accountable for the firm position he takes up is difficult to decide. If this ignorance be so important as Mr Cantor thinks, then it must count for a good deal in the judgment we form as to the bitter feelings that Newton and his supporters entertained. As regards the later developments it should be noted that Newton in his letter to Conti of February 26th, 1716, expressly deduces from a remark in the first letter of Leibnitz to Conti the conclusion that Leibnitz had seen the second letter at Collins's. However improbable it is that Leibnitz then saw the letter, he gives no contradiction to Newton's assertion; though in justice to Leibnitz it should be added that he refused to discuss the details of letters till he had

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apparently declined to take action. On the 6th of March the Society, "since Leibnitz had appealed from Keill as *homine novo* to the Society itself," ordered that the older documents should be consulted, and appointed certain Fellows, who seemed more competent to examine them, to inquire into the matter and to report what they found in the old writings, along with their opinion." At first the Committee consisted of Arbuthnot, Hill, Halley, Jones, Machin and Burnet, but later Robartes, Bonet (the Prussian minister), De Moivre, Aston and Brook Taylor were added.

Mr Cantor discusses the composition of the Committee and recognises the competence of Halley, Machin, and Jones; also of De Moivre and Brook Taylor, but as they were only chosen a week before the Report was submitted, he considers their presence as merely ornamental. There is no evidence, he thinks, that the others were fit to judge in the case.* It is, by the way, only by straining the meaning of the words that the Committee can be described as composed of "gentlemen of various nations," as Newton maintained it was.

On April 24th the Report was submitted. Its main points are:—I. Leibnitz's visits to London and correspondence with Collins; Collins's liberality in communicating what he had from Newton and Gregory. II. Leibnitz's appropriation of Mouton's differential method and the want of evidence that he had any other before his letter of 21st June 1677, a year after the tangent letter had been sent to Paris to be communicated to him, "in which letter the method of fluxions was sufficiently described to any intelligent person." III. Evidence that Newton had invented his method before 1669. IV. The differential method is one and the same with the method of fluxions, excepting the name and mode of notation; the attribution to Leibnitz of the calculus was due to ignorance of the correspondence with Collins and Oldenburg. The Committee reckon that Newton was the first inventor, and Keill, in asserting the same, had been no ways injurious to Leibnitz; and they recommend the publication of documents. †

* De Morgan says Burnet was a son of the Bishop and a pupil of John Craig (*Phil. Mag.* [4 S] IV., p. 325, Note † (1852).

† Mr Cantor has misunderstood the last clause of the Latin version of the Report; it obviously means, "as also whatever else fit for elucidating this history might occur in the A.E."

The Society accepted the recommendation and undertook the cost of publication, though it did not formally assent to the judgment of the Committee. The collection of documents, now known as the *Commercium Epistolicum*, was sent in name of the Society to mathematicians in various countries, and, later on, 25 copies were directed to be sent to a bookseller in Holland, to be disposed of by him for behoof of the Society. The Society apparently took no special pains to let Leibnitz have a copy without delay.

The most important documents in the book are:—The *De Analysi;* letters between Newton and Collins, between Collins, Gregory and other mathematicians, between Newton, Oldenburg and Leibnitz, between Wallis and Leibnitz; extracts from Fatio's *Lineae brev. desc.*, from the review of the *De Quadratura;* the letters between Leibnitz and Sloane, between Keill and Sloane; and finally the Report of the Committee, in the original English and in a Latin translation where "to any intelligent person" appears as "*idoneo harum rerum cognitori.*" The original English is probably the rashest assertion ever made on a mathematical subject, and the Latin version is certainly needed to interpret it. Apparently the Committee attributed more to the intelligence of the English readers of the Report than they did to that of the foreign mathematicians.

It would be too much to expect that the Committee should not have strong leanings in favour of Newton, and had they confined [redacted]

Leibnitz, are not, I think, unjust; but to any one reading the book, the cumulative impression of the notes and comments on the work of a man who, it must be remembered, had had no chance of giving his own version, would be all but certainly one of his complete depravity.

It is difficult to make out who actually drew up the foot-notes and the connecting narrative, for the original English Report only speaks of Extracts from Letters and Papers as being presented to the Society; Halley, Jones and Machin seem to have acted as editors of the *Commercium*, but the book was ordered to be delivered to each member of Committee to examine it before its publication. A note appended to the passage of Leibnitz's letter to Sloane, in which he refers to Newton, reads, "Newton and Leibnitz are neither fit judges nor witnesses. On the old documents the judgment must be based." The natural inference from this note is that the Committee had consulted Newton just as little as Leibnitz. But this inference would not be correct, for one portion at least of the connecting narrative (Cantor, p. 300) is from a manuscript of Newton, drawn up by him, in all probability, about the beginning of 1712 and therefore not an "old document." If Newton had merely placed his old MSS. at the disposal of the Committee there would be little fault to find with the foot-note translated above, though it would be unfair not to have requested the same information from Leibnitz; but Newton's manuscript was written specially in view of the controversy, and the foot-note directly suggests what the Committee knew to be not true.

I do not wonder that Newton felt deeply insulted by the terms of Leibnitz's paper in the *Acta*, for in the two Oldenburg letters he had, while refraining from any exposition of his method of fluxions, been exceedingly generous in his communications; and whether Leibnitz gained much or little from them for his calculus, his later work must have convinced him that Newton had anticipated him on many points. Indeed before the Fatio incident Leibnitz had frequently stated as much. But however much I sympathise with Newton (if I may venture to use such a phrase) I cannot see that his conduct in the controversy raised by Keill is at all worthy of him; his claims were, in my judgment, so indisputable that he might well have put the most generous construction on the work of Leibnitz and have passed by his insinuations with the contempt

they deserved. And besides, the keenness with which he felt the insinuations of Leibnitz might have led him to see how Leibnitz must have suffered under the charges of Fatio—accusations that Leibnitz refuted by reference to Newton's own published statement, interpreted in a way that Newton did not challenge.

The controversy that followed on the publication of the *Commercium Epistolicum* is far from pleasant reading. Leibnitz had good grounds of complaint, and his multifarious occupations left him little time or opportunity for effective reply. He was for a considerable time absent from Hanover, and he refused to discuss details of dates and documents till he had an opportunity of examining his papers. But he only increased the suspicions attaching to him by his wild charges against Newton and by the introduction of matter wholly foreign to the dispute. The *Charta volans* of 1713, with its extracts from the letter of an eminent mathematician, merely made matters worse for him, and the mediation of Chamberlayne and Conti was of no avail. Mr Cantor in referring to Newton's charge that Leibnitz had borrowed from Barrow (p. 311) does less than justice to Leibnitz's acuteness; Leibnitz answers Newton in practically the same terms as Mr Cantor does himself. (Letter to Conti of 9th April 1716.) The letters produced by the mediation contain passages in which Leibnitz, with justice, insists on the unfairness of charging one with plagiarism who has developed theories beyond the stage which the original authors reached, but [redacted]

abroad. The narrative is written from the point of view of the foot-notes, and contains one assertion that is not explicitly made in the *Commercium* itself, namely, that Leibnitz saw Newton's second letter on his second visit to London. Such an assertion might have some reasonableness after Newton's letter to Conti of 26th February 1716, but there are no grounds apparent for such a positive statement at the time when the "Account" was written. It seems as if the assertion can have been at best only a guess; and it was grossly unfair, to say the least, to introduce into what professed to be an account of the *Commercium* an important piece of evidence like this without express mention that it was new and with proof of its truth. The "Account" states in positive terms that the Society could only proceed, in the way of answer to Leibnitz's letter, by an examination of old documents, and could not accept either Newton or Leibnitz as a witness. But it has been pointed out above that the Committee quoted from one recent manuscript of Newton, and while their actual quotation occurs in the connecting narrative supplied by the editors and is not presented as a paper in the case, it nevertheless inevitably suggests that Newton had more to do in the preparation of the *Commercium* than the Committee give its readers to understand. The suggestion can only be confirmed by the knowledge, which is now certain, that the author of the *Account* in the Transactions of the Society was Newton himself.

In 1716 Leibnitz died without publishing his own *Commercium*, and in 1722 a new edition of the *Commercium* of 1712 was issued, with a new preface, a Latin translation of the "Account," and an Appendix containing the opinion of the eminent mathematician quoted in the *Charta volans* of 1713 with an *Annotatio* on this opinion. With the exception of these additions, the new issue is represented as a reprint of that of 1712, retaining even the original title page before the older portions. But it is as a matter of fact not a simple reprint; there are several changes, most of them of no great importance, though unwarrantable according to modern notions, in a professed reprint; but one addition has reference to a document (the tangent letter of 1672) that has a prominent place in the Committee's Report. The *Collectio* of Collins which contained that letter in full is now said to have been sent to Leibnitz on 26th June 1676. Recent inquiries have made it all but certain that the *Collectio* was never sent but only an abridgement which

did not contain the example by which the method of drawing tangents, as the *Commercium* (p. 100) says, was expounded.

The author of the new introduction and of the *Annotatio* was Newton. The opening sentence of the *Annotatio* might be made to bear the interpretation put on it by M. Lefort in his edition of the *Commercium*, namely, that the writer of it suggests that Keill was the author of the *Recensio*; it is possible to put another interpretation on it, but no interpretation is likely to make this part of the case much better for Newton. Sir David Brewster, in his *Life of Newton*, speaks thus:—"It is due to historical truth to state that Newton supplied all the materials for the *Commercium Epistolicum* and that, though Keill was its editor and the Committee of the Royal Society the authors of the Report, Newton was virtually responsible for its contents."

Mr Cantor remarks in his closing words on the controversy that it had no proper continuation and that till the present century the conviction of Leibnitz's plagiarism from Newton was all but universal. The publication of Leibnitz's papers has changed the attitude of the mathematical public; though whether that change is so complete as Mr Cantor says, is doubtful. The adherents of Newton were right in contending that Newton was first in possession of the calculus; it is equally true that it was Leibnitz who first made it accessible to mathematicians. How much Leibnitz derived from Newton can never be definitely settled; I think all the evidence shows that he was on the lines of his calculus before

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Edinburgh Mathematical Society.

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JAMES TAYLOR, M.A., The Institution, Dollar.

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130 William Welsh, M.A., Jesus College, Cambridge.

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The following Presents to the Library have been received, for which the Society tenders its grateful thanks.

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1. Proceedings of the London Mathematical Society.
   Vol. XXVI.—Nos. 509-534.
   Vol. XXVII.—Nos. 535-548.

2. Proceedings of the Royal Society of Edinburgh.
   Vol. XX.

3. Memoirs and Proceedings of the Manchester Literary and Philosophical Society.
   Vol. X.—Nos. 1, 2, 3.

4. General Reports of the Association for the Improvement of Geometrical Teaching, for the years 1882, 1883, 1887-93.

5. Mathematical Gazette.
   Nos. 1-8.

6. Reprint of Napier's "Constructio" from Lyons Edition of 1620.
   (Presented by the Publisher, M. Hermann, Paris.)

7. Indian Meteorological Memoirs.
   Vol. VII.—Parts 1-5.
   Vol. VIII.—Part 1.
   (Presented by the Royal Society of Edinburgh.)

8. Bulletin of the American Mathematical Society.
   Second Series—Vol. I.—No. 10.
   Vol. II.—Nos. 1-10.

9. The Kansas University Quarterly.
   Vol. IV.—Nos. 1, 2, 3, 4.

10. Mathematical Magazine.
    Vol. II.—No. 10.

11. Bulletin de la Société Mathématique de France.
    Vol. XXIII.—Nos. 6-10.
    Vol. XXIV.—Nos. 1-4.

12. Annales de la Faculté des Sciences de Marseille.
Vol. V.—Parts 1-4.
Vol. VI.—Parts 1-3.

13. Journal de Mathématiques Élémentaires.
Publié par M. H. Vuibert.
Nos. 1-20. 1895-96.

14. Journal de Mathématiques Élémentaires.
Publié sous la direction de M. de Longchamps.
Nos. 8-12. 1895.
Nos. 1-7. 1896.

15. Journal de Mathématiques Spéciales.
Publié sous la direction de M. de Longchamps.
Nos. 8-12. 1895.
Nos. 1-7. 1896.

16. Note sur la machine a résoudre les Équations de M. Torres.
By M. Maurice D'Ocagne.

17. Jornal de Sciencias Mathematicas e Astronomicas.
Publicado pelo Dr F. Gomes Teixeira.
Vol. XII.—Nos. 3, 4, 5.

18. Revue Semestrielle des Publications Mathématiques.
Published under the auspices of the Mathematical Society of Amsterdam.
Vol. III.—Part 2.
Vol. IV.—Part 1.

[redacted]

Fig. 1

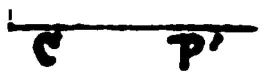


Fig. 2

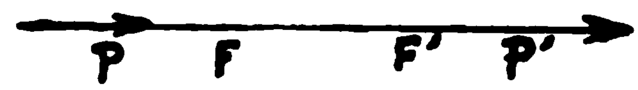
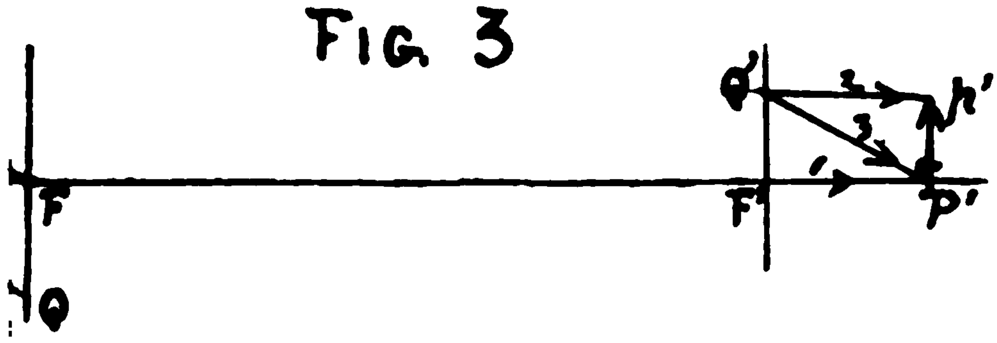


Fig. 3



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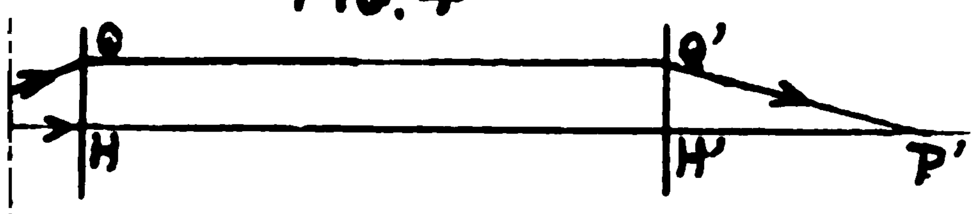


Fig. 5

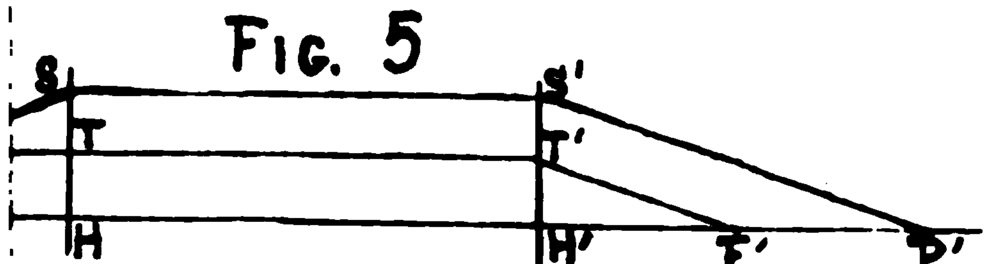
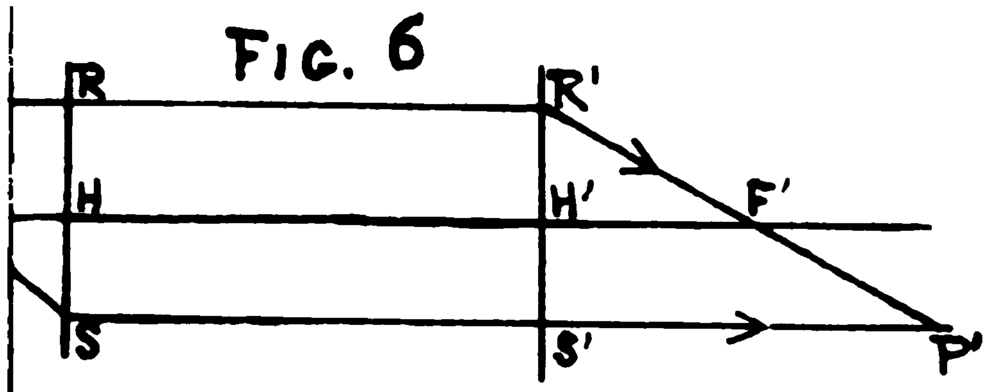


Fig. 6



12. Annales de la Faculté des Sciences de Marseille.
Vol. V.—Parts 1-4.
Vol. VI.—Parts 1-3.

13. Journal de Mathématiques Élémentaires.
Publié par M. H. Vuibert.
Nos. 1-20. 1895-96.

14. Journal de Mathématiques Élémentaires.
Publié sous la direction de M. de Longchamps.
Nos. 8-12. 1895.
Nos. 1-7. 1896.

15. Journal de Mathématiques Spéciales.
Publié sous la direction de M. de Longchamps.
Nos. 8-12. 1895.
Nos. 1-7. 1896.

16. Note sur la machine a résoudre les Équations de M. Torres.
By M. Maurice D'Ocagne.

17. Jornal de Sciencias Mathematicas e Astronomicas.
Publicado pelo Dr F. Gomes Teixeira.
Vol. XII.—Nos. 3, 4, 5.

18. Revue Semestrielle des Publications Mathématiques.
Published under the auspices of the Mathematical Society of Amsterdam.
Vol. III.—Part 2.
Vol. IV.—Part 1.

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Fig. 1

Fig. 2

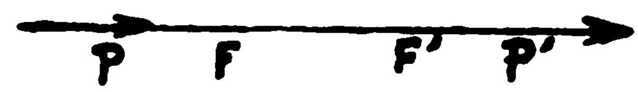
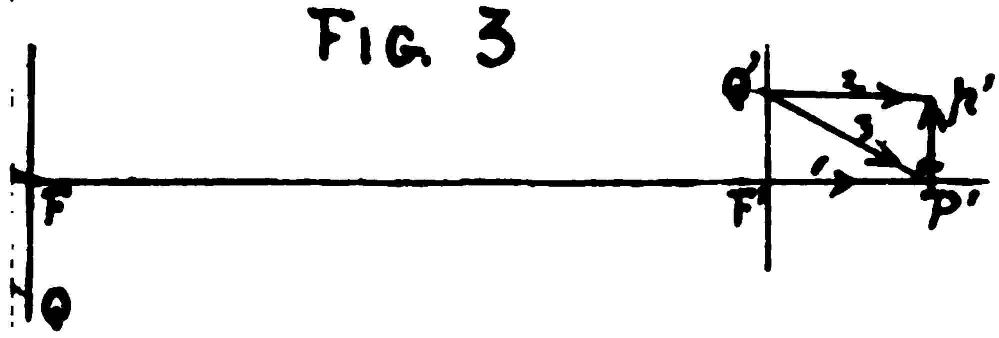


Fig. 3



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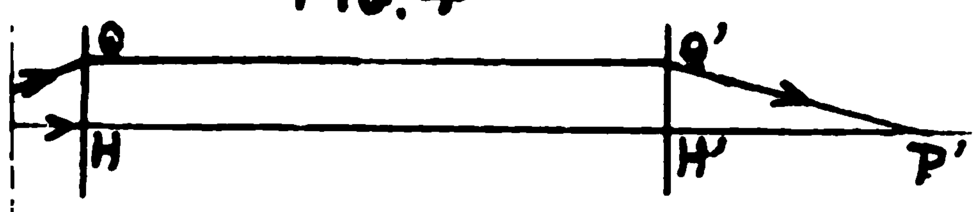

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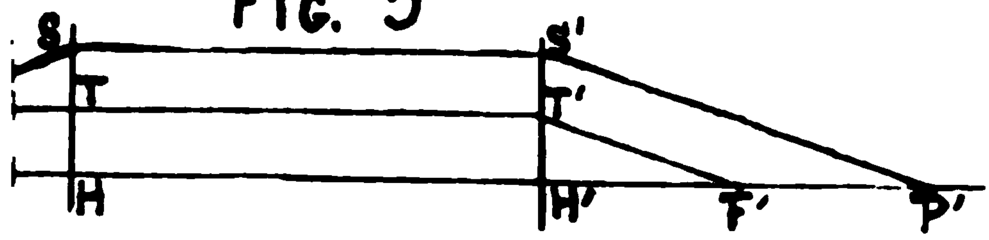

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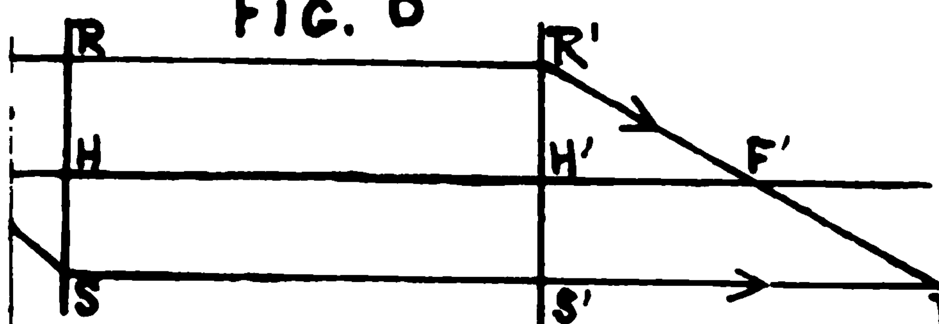

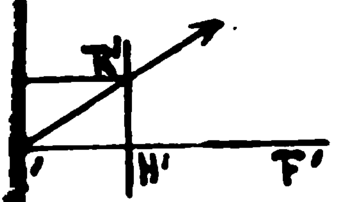

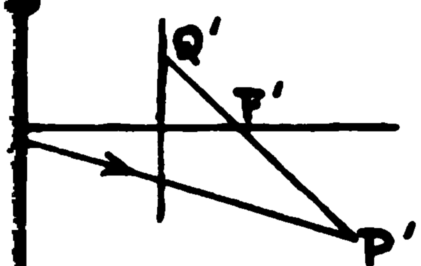

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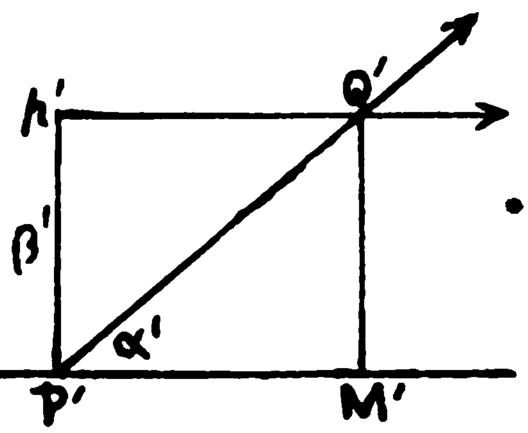

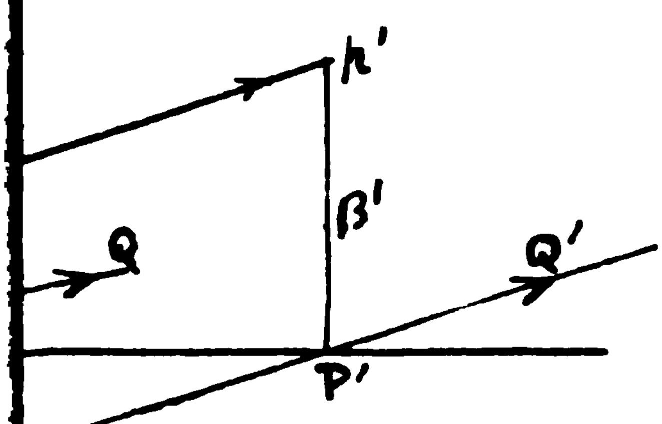

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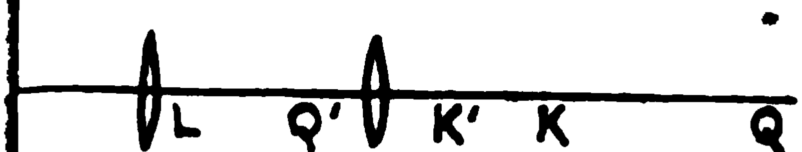

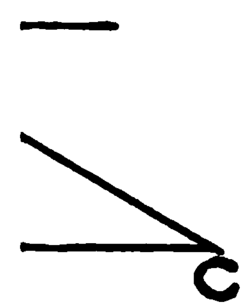

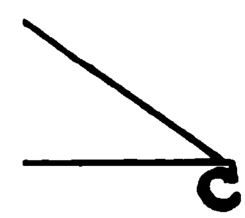

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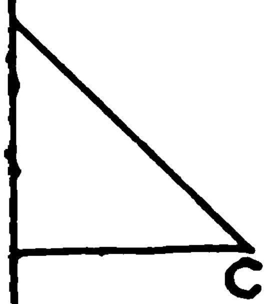

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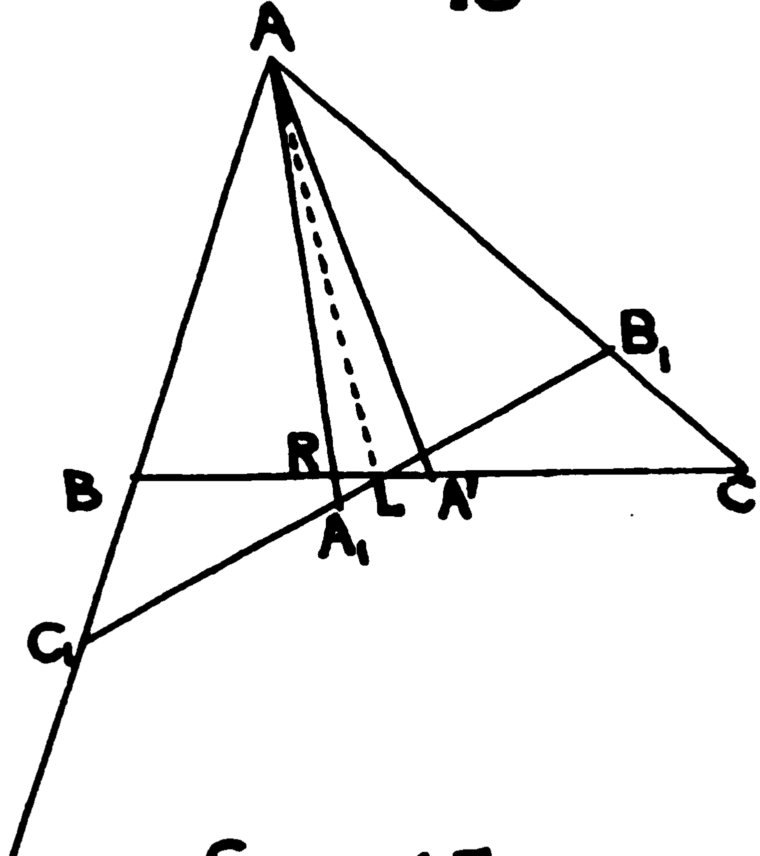

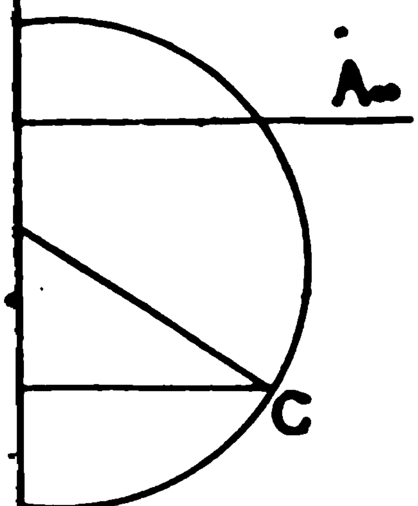

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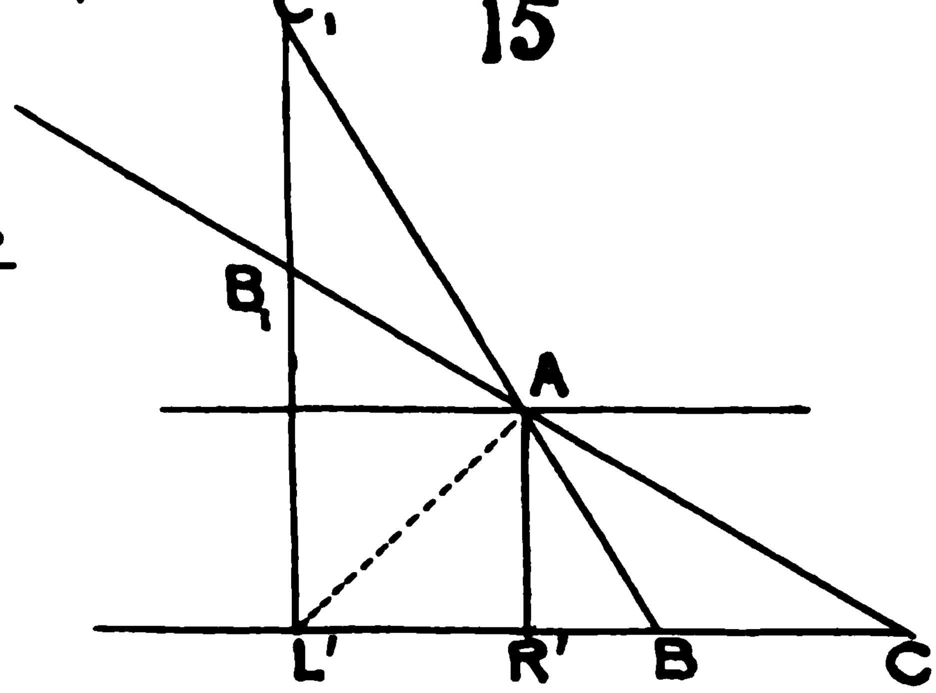

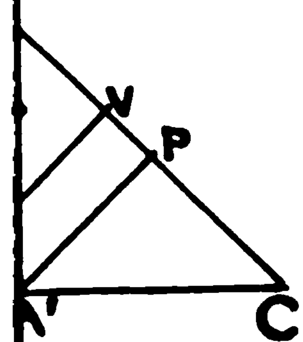

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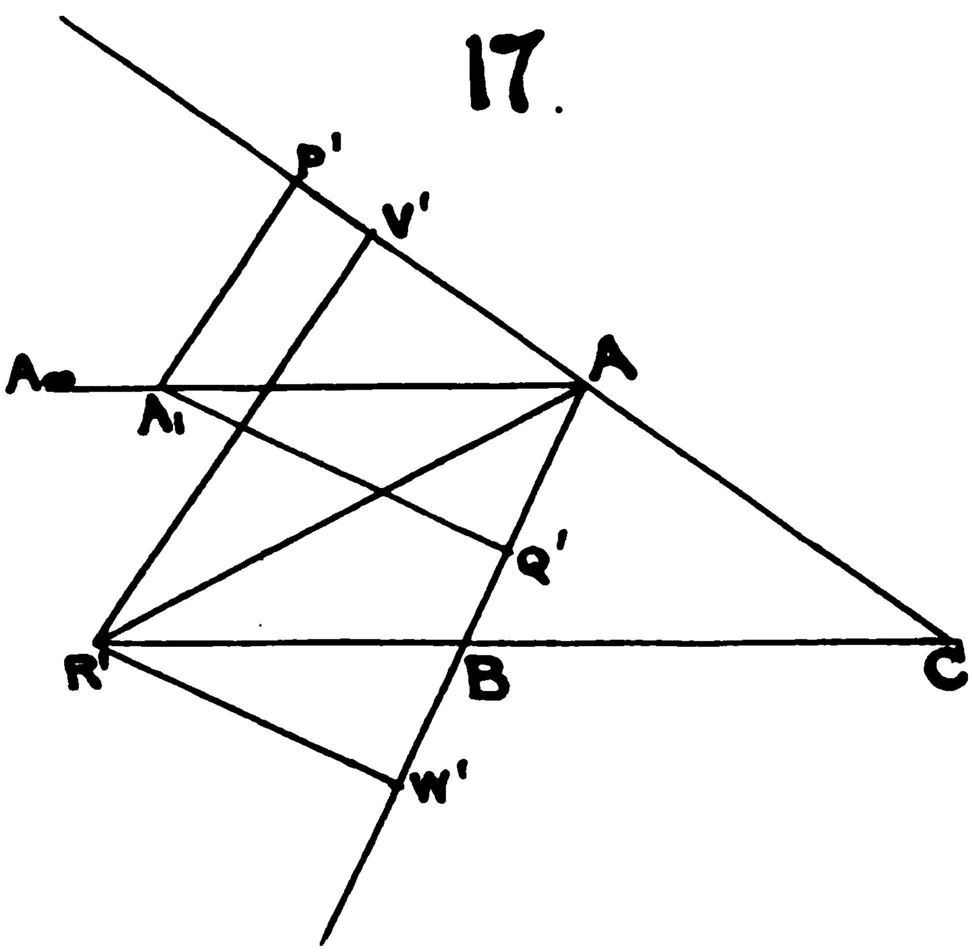

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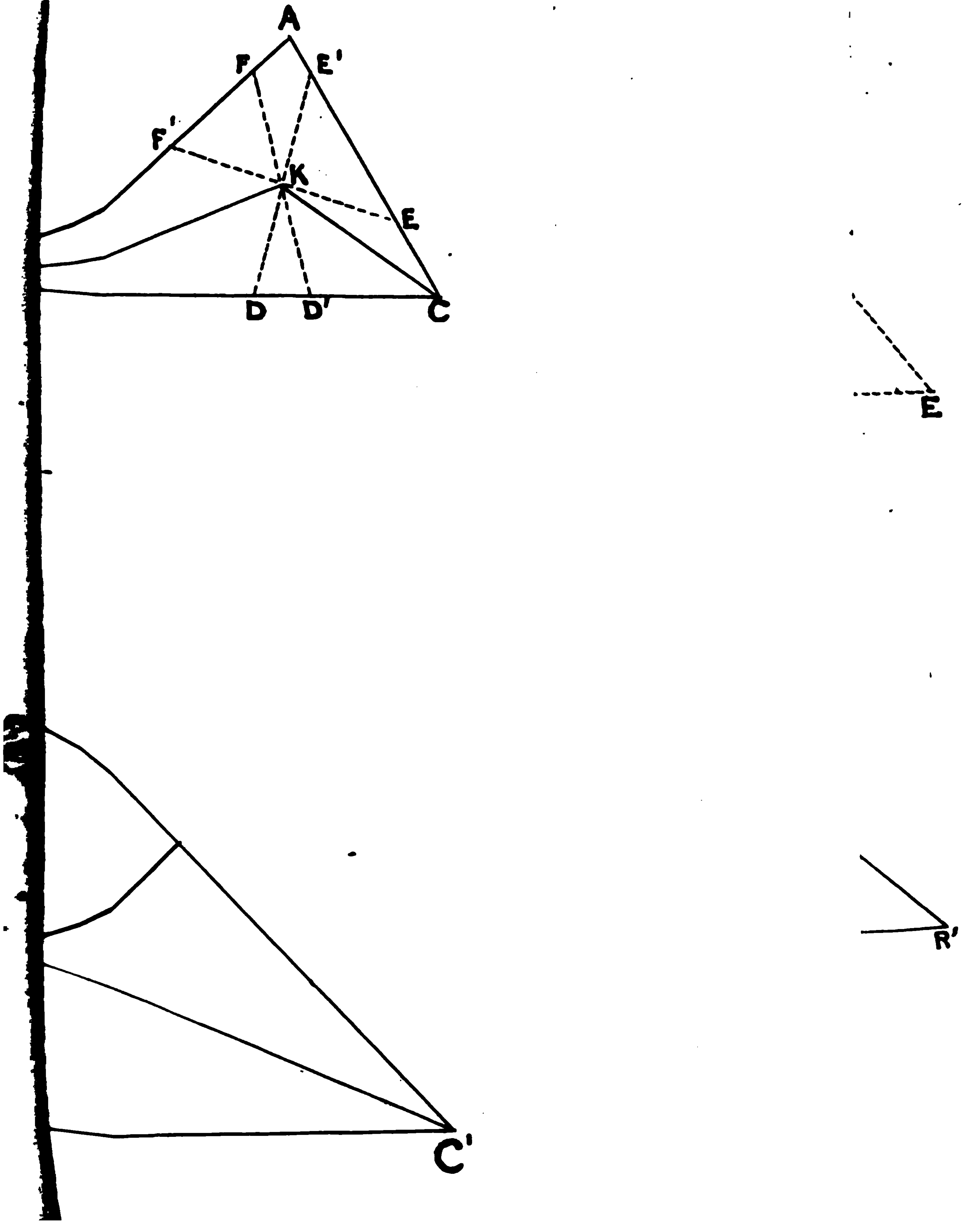

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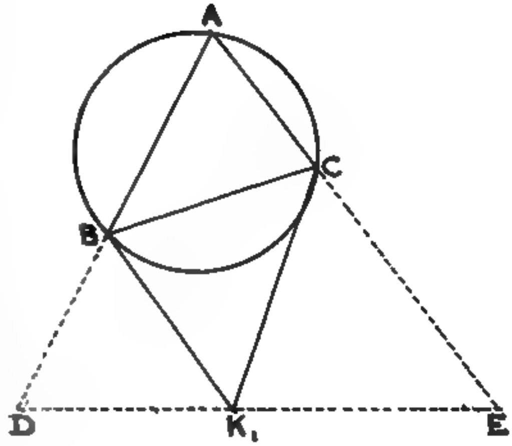

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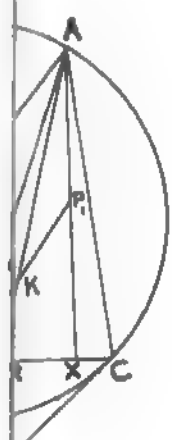

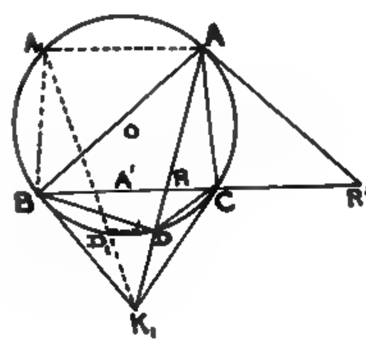

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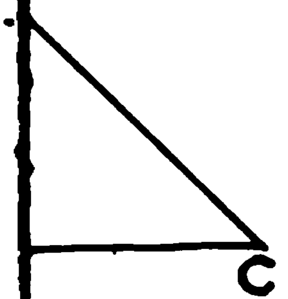

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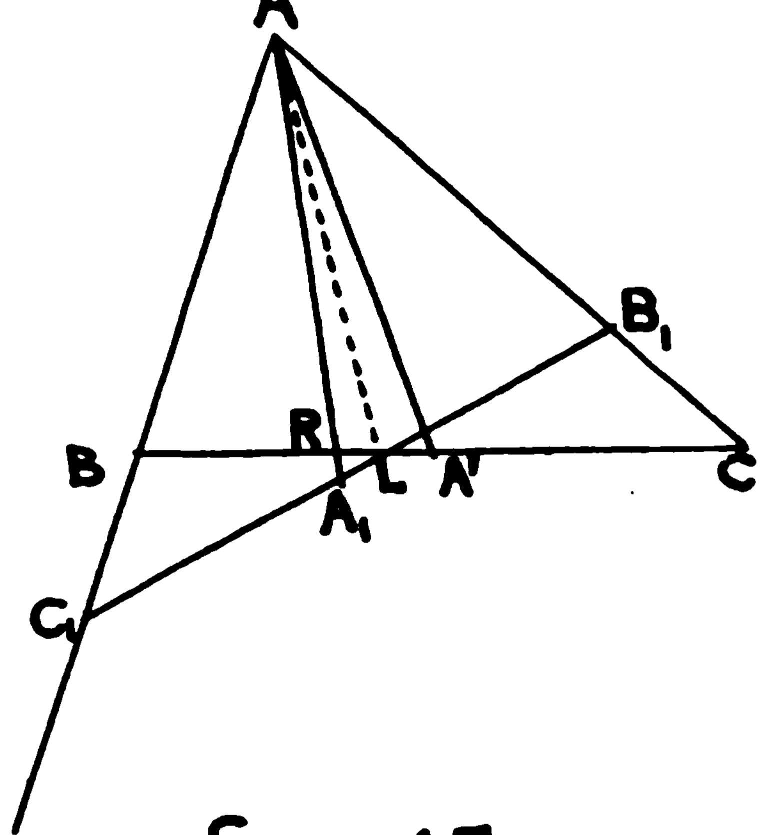

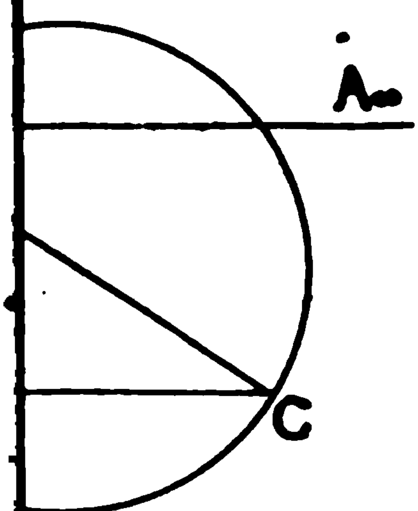

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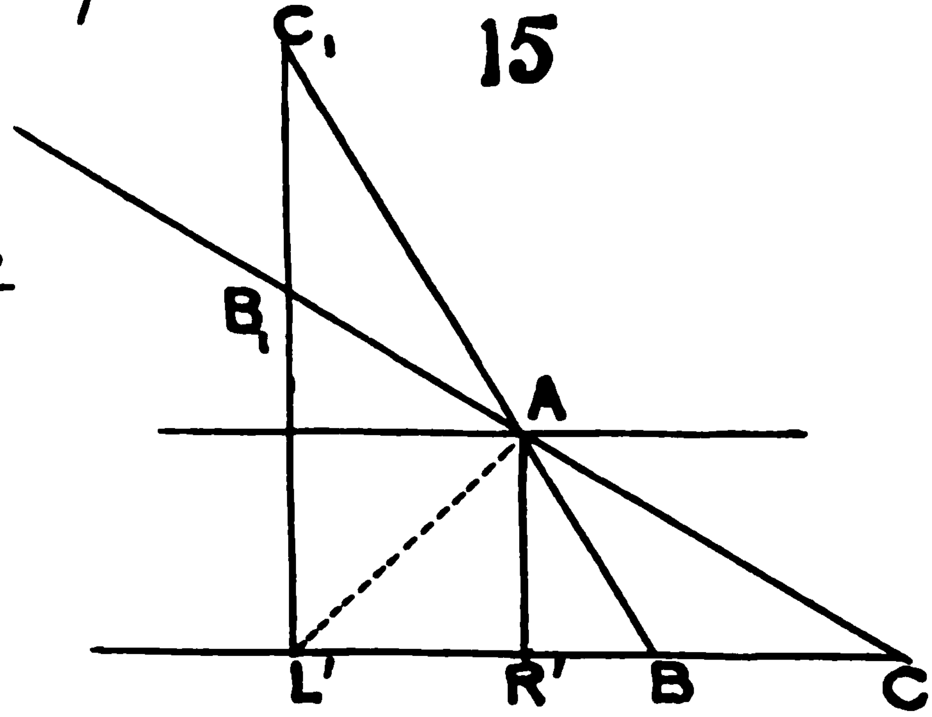

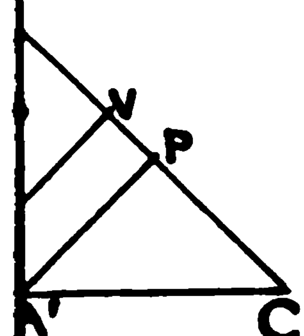

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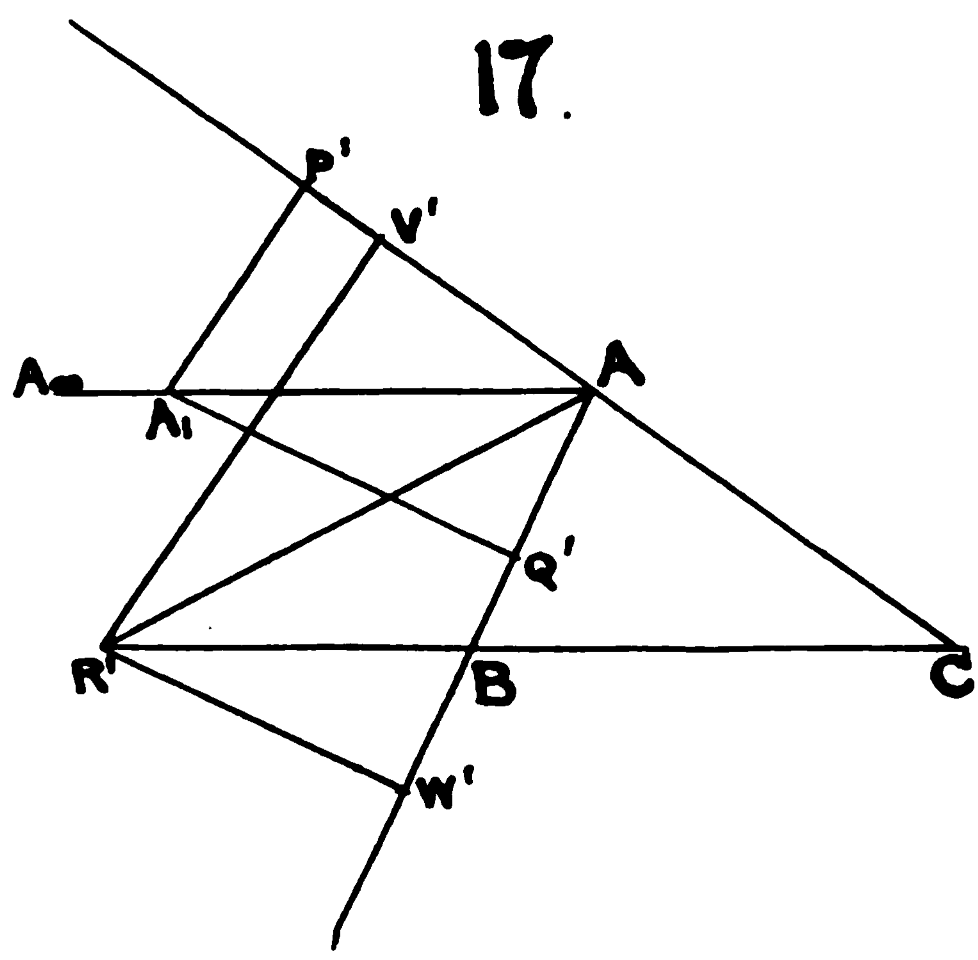

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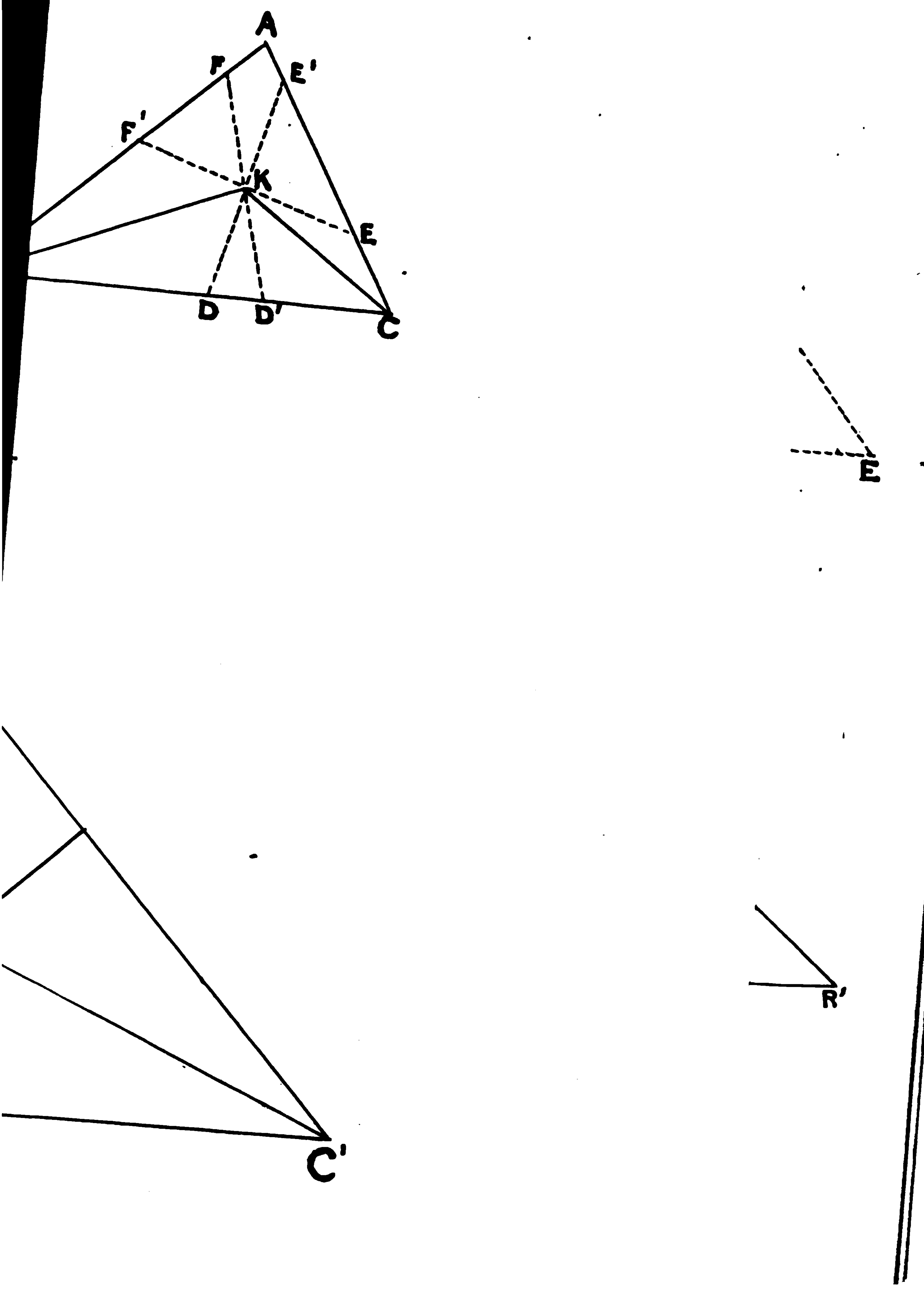

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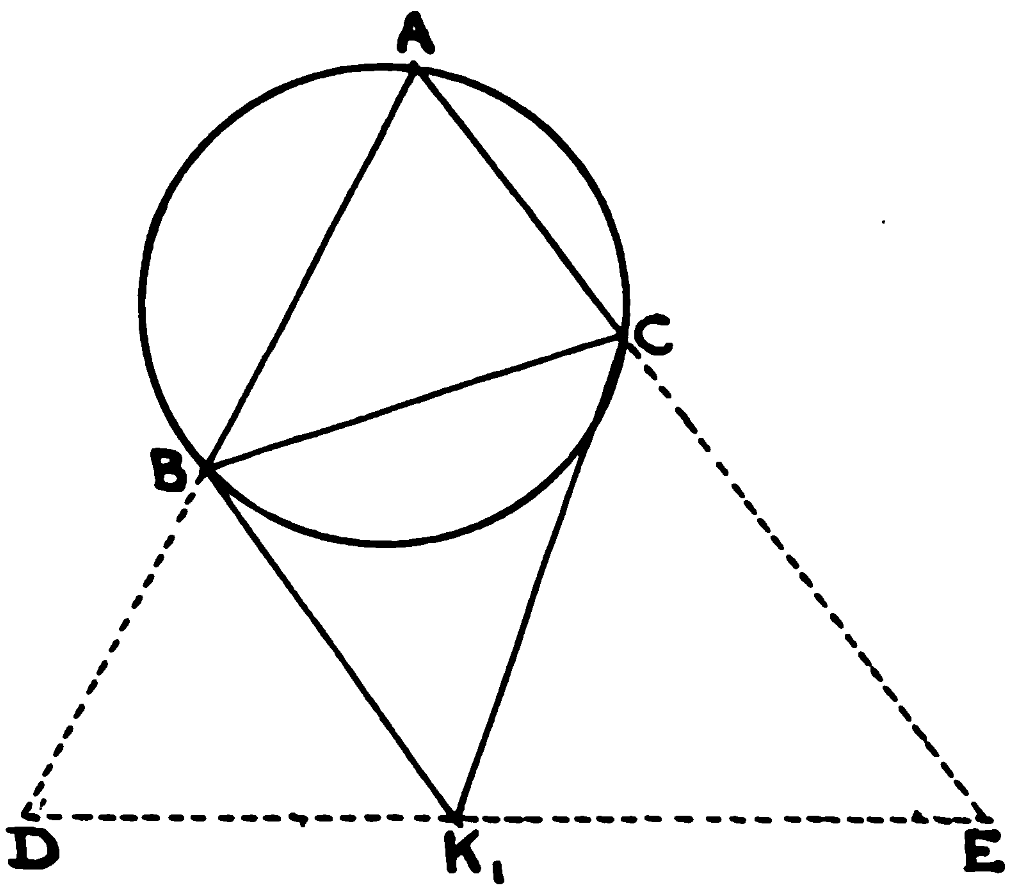

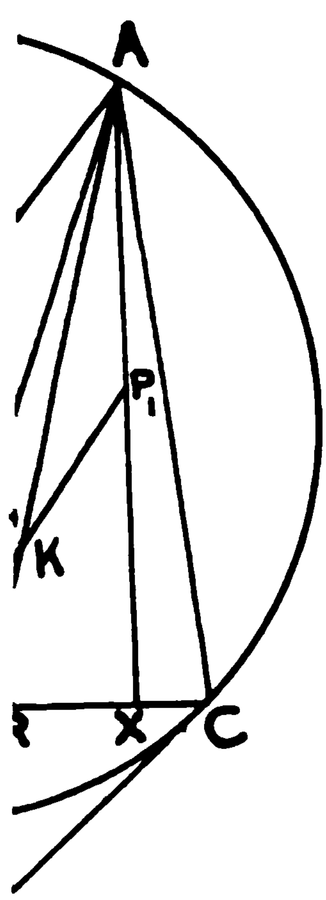

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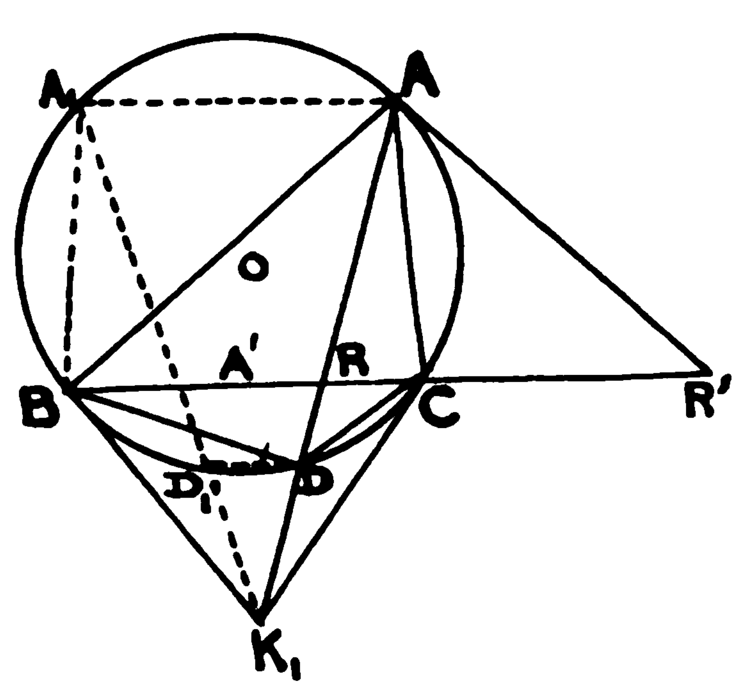

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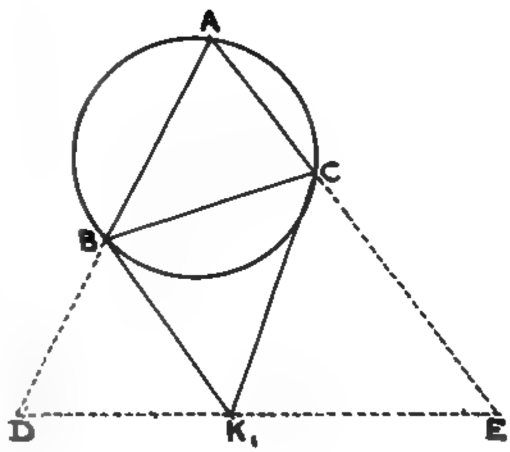

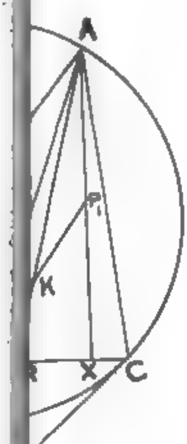

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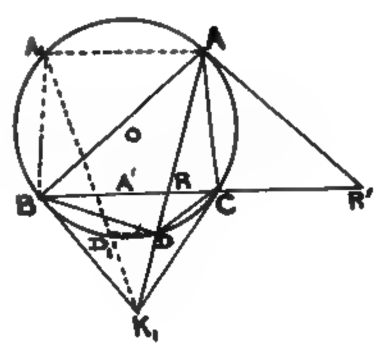
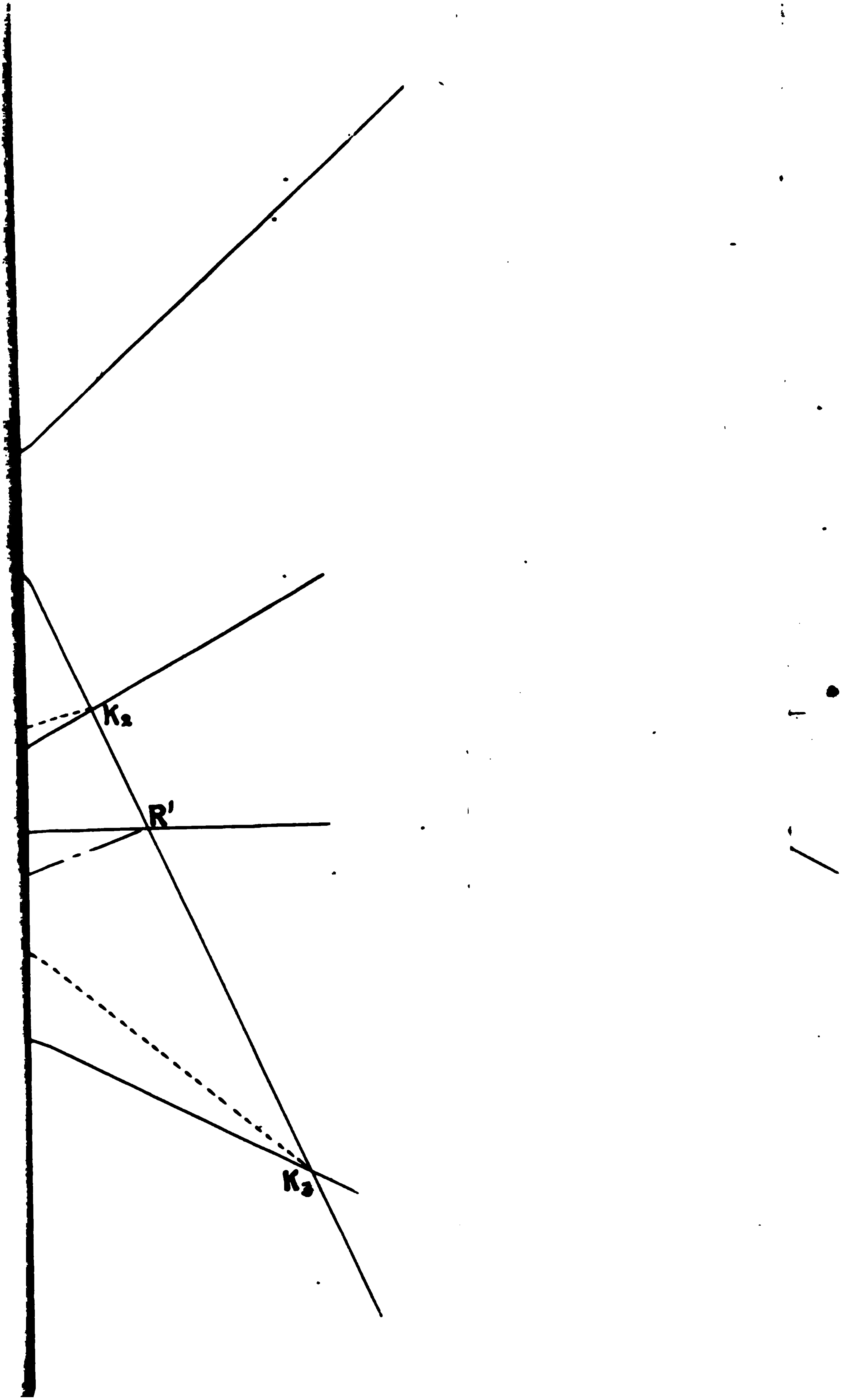
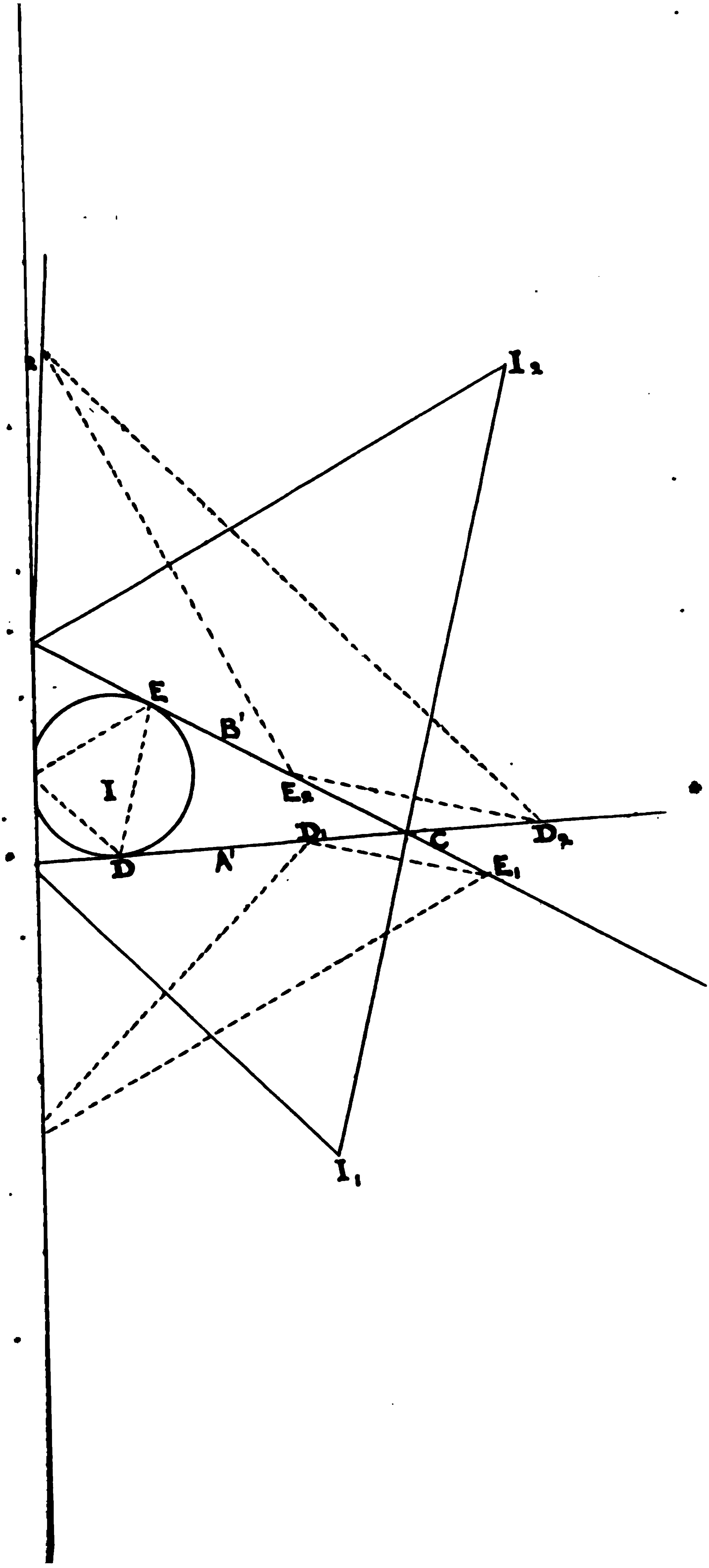
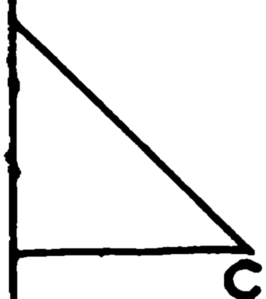



K_2
R'
K_3



I₂
E
B'
I
E₂
D₁
C
D₂
D
A'
E₁
I₁

12



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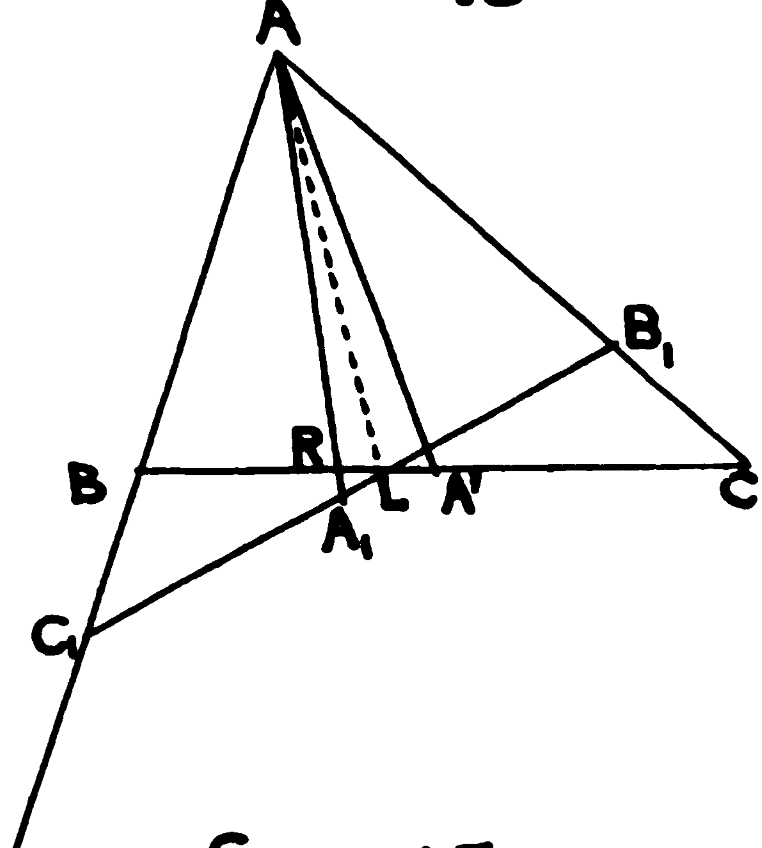

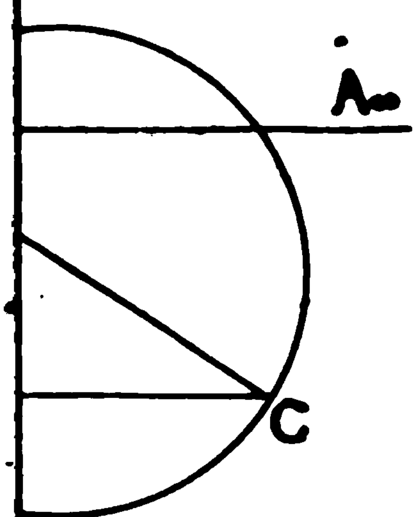

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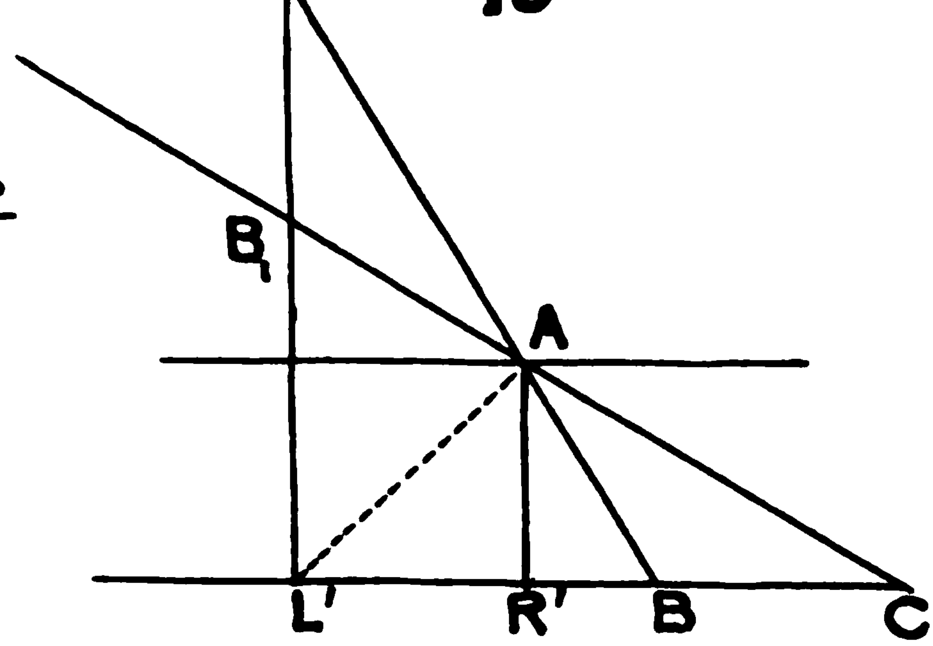

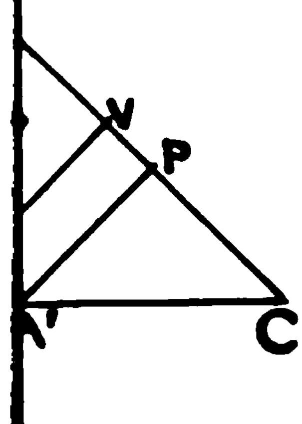

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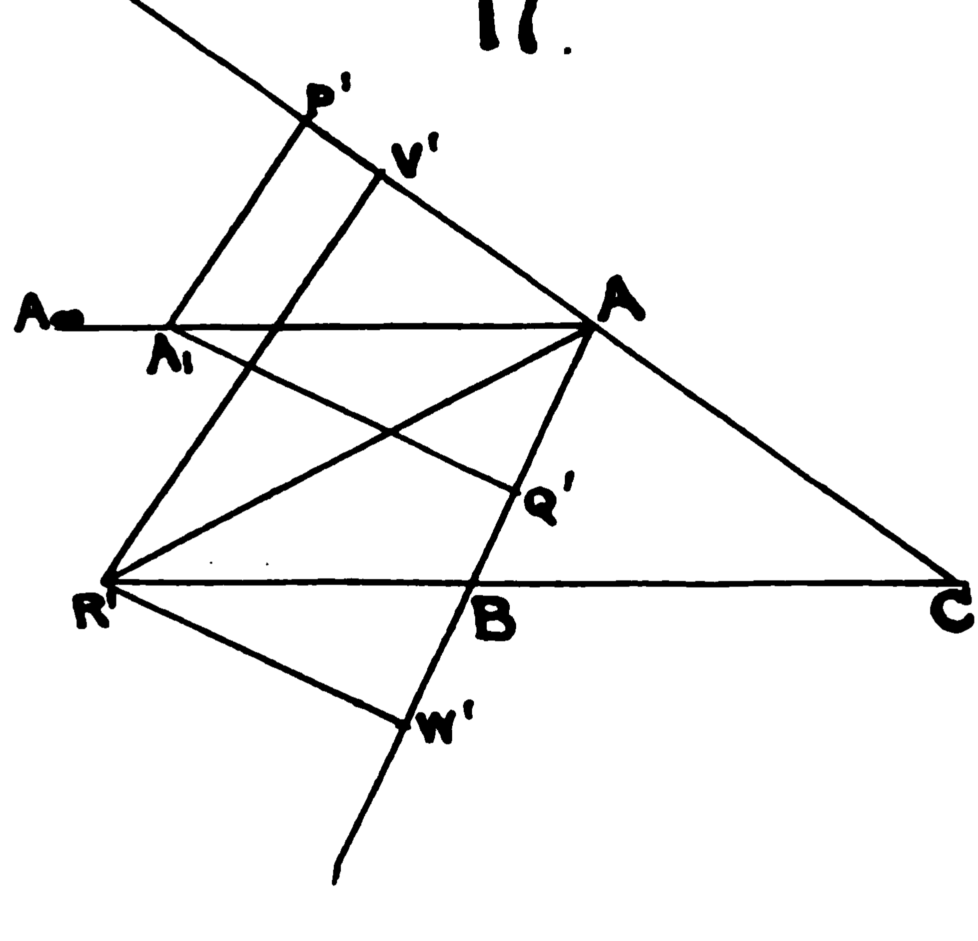

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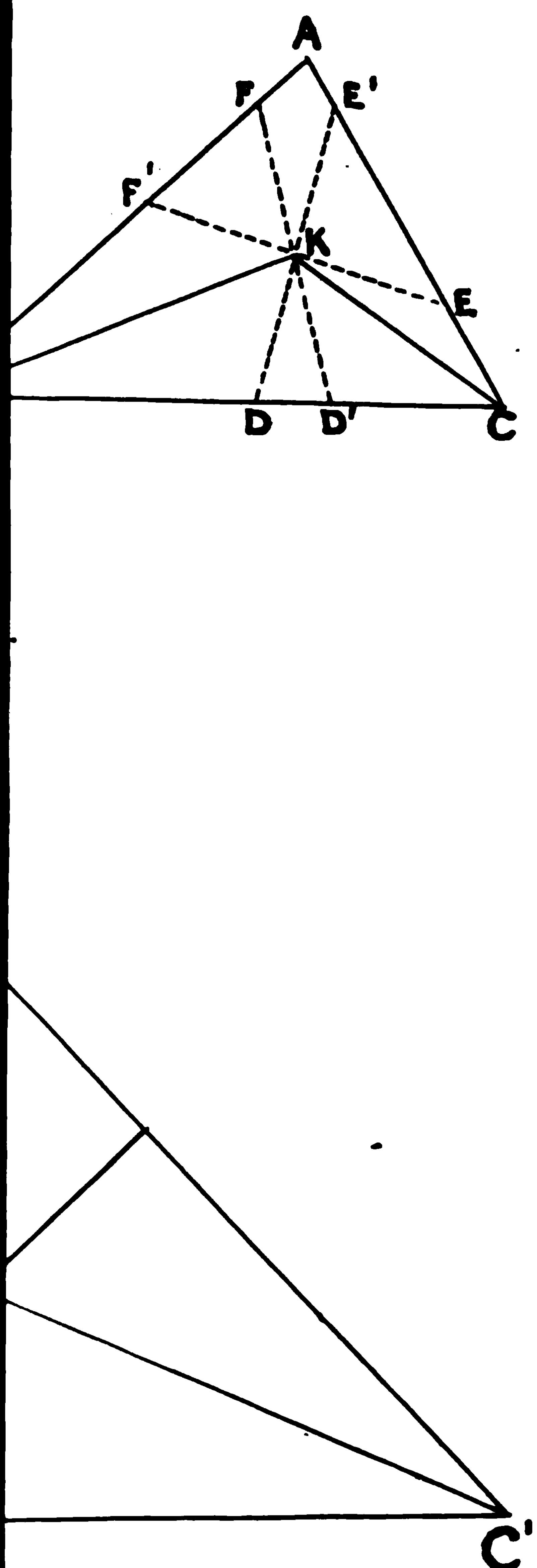

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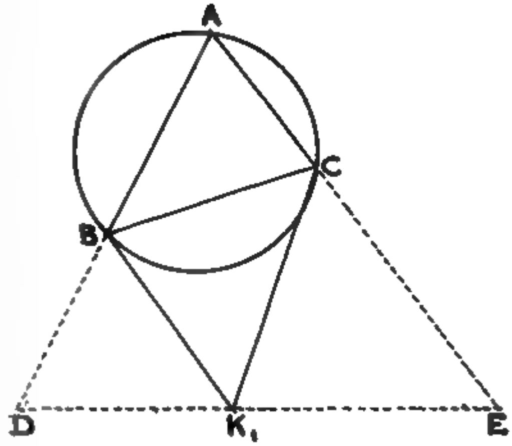

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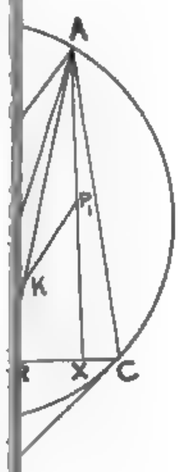
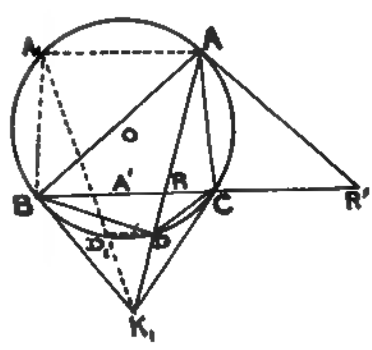
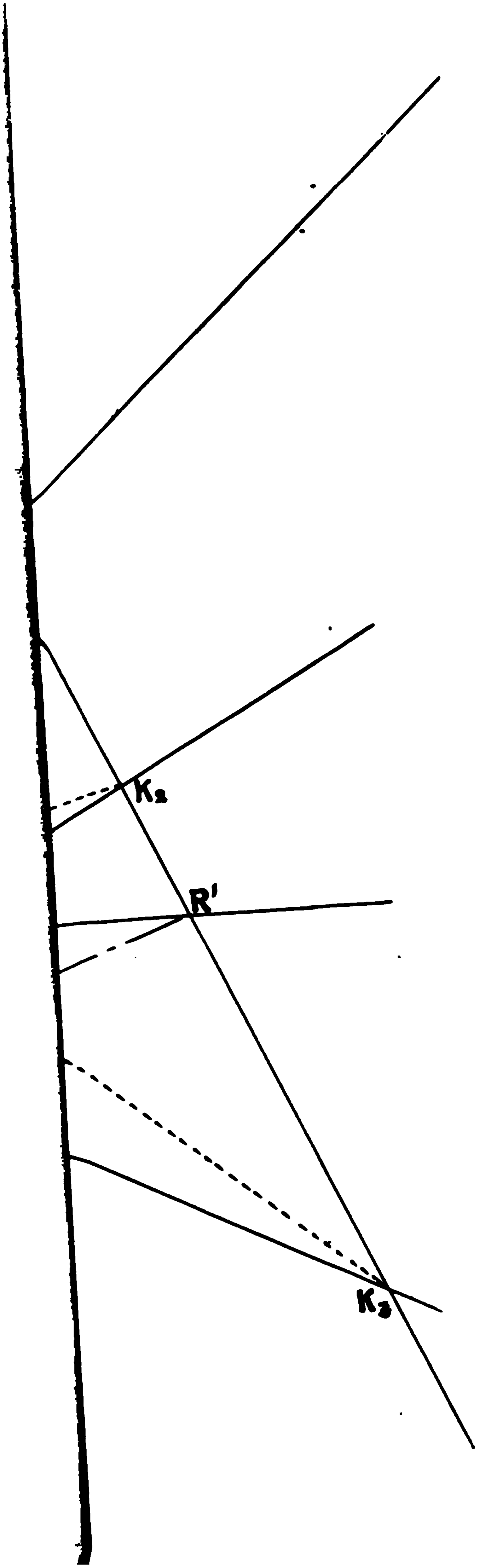
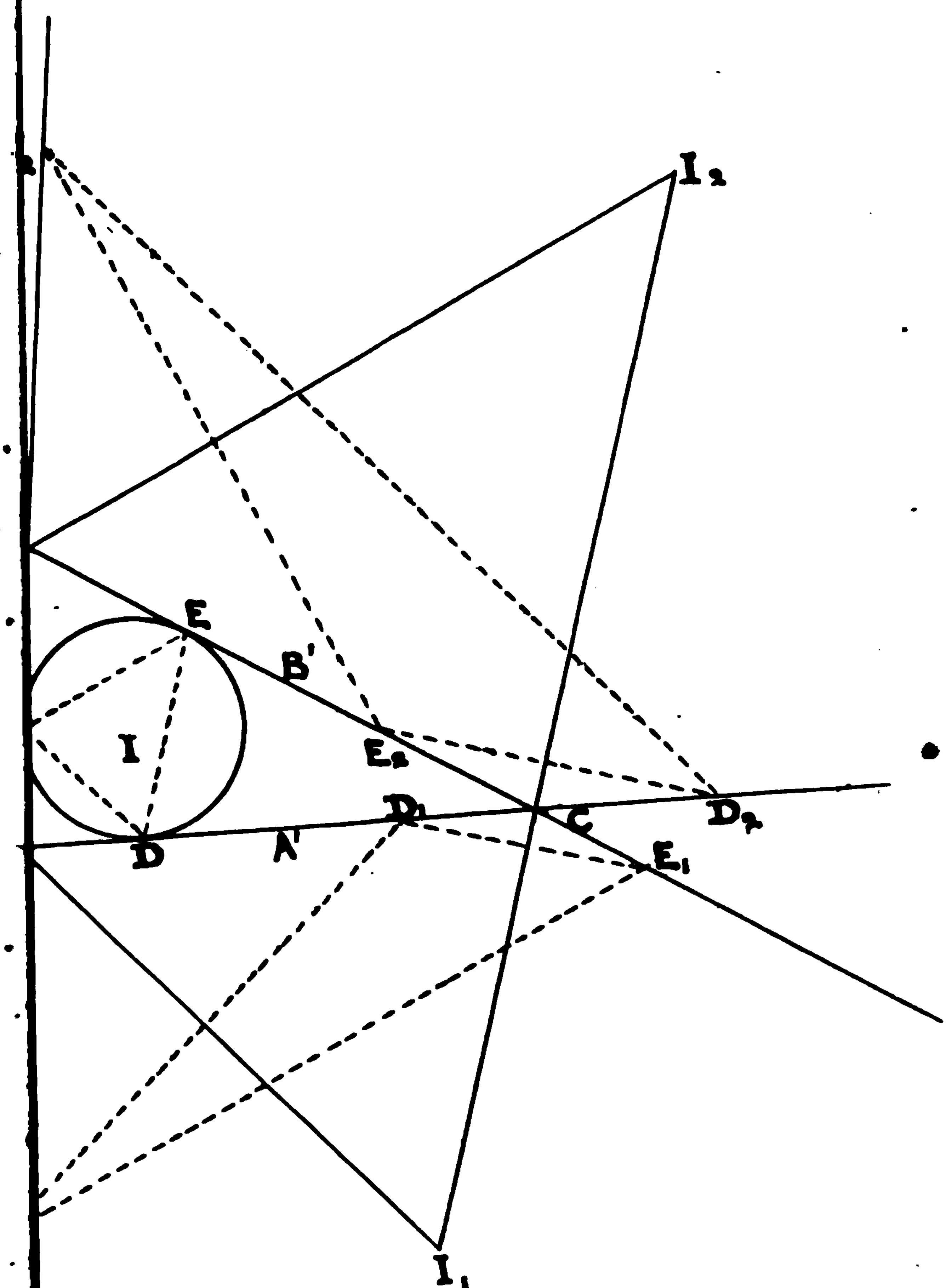


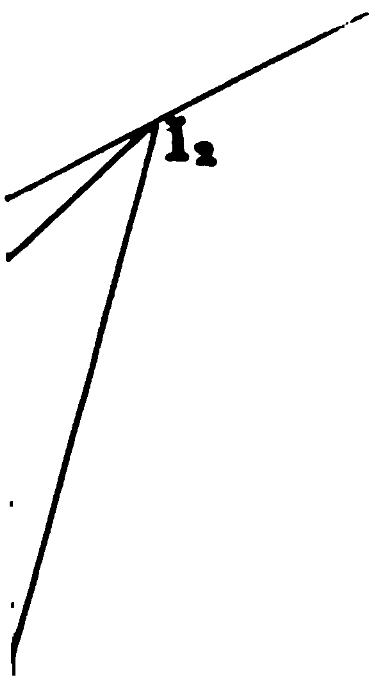


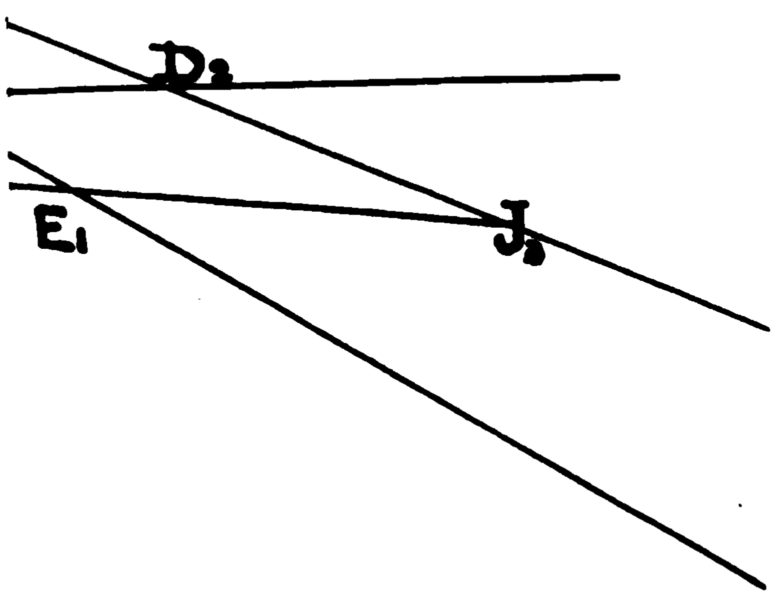


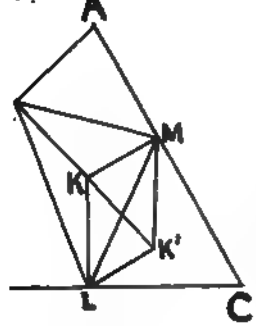
K₂
R'
K₃



I_2
E
B'
I
E_2
D_1
C
D_2
D
A'
E_1
I_1


I_2


D_2
E_1
J_2



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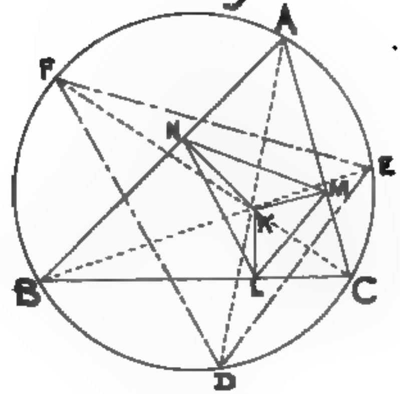

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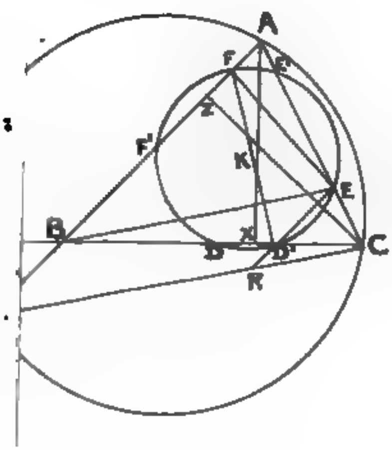

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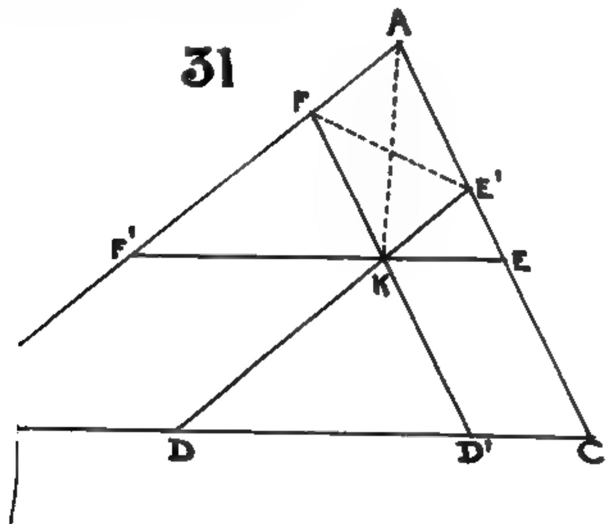

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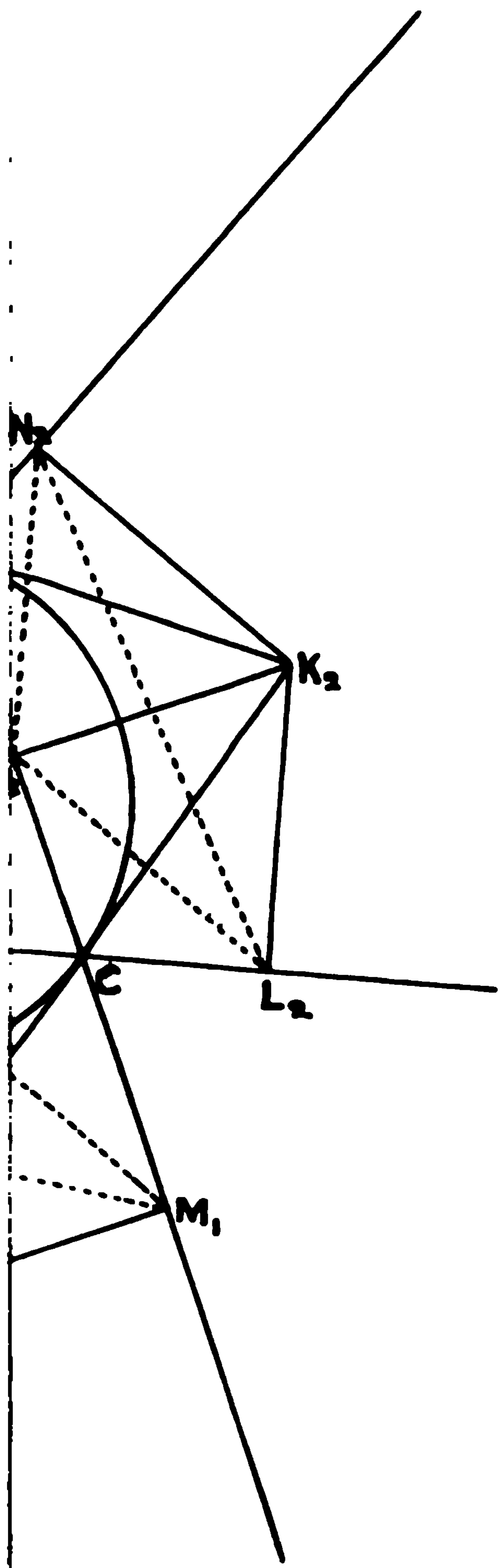
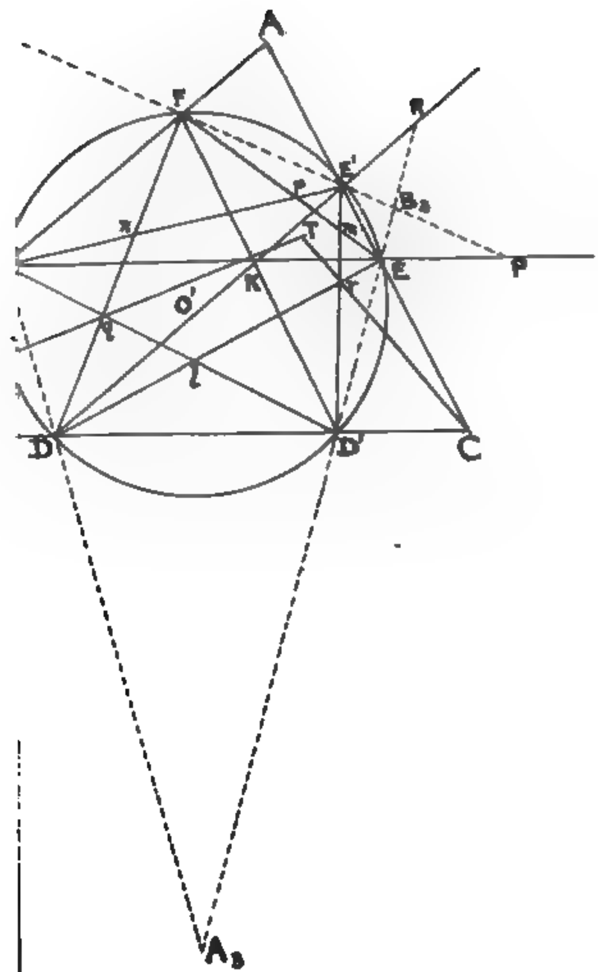
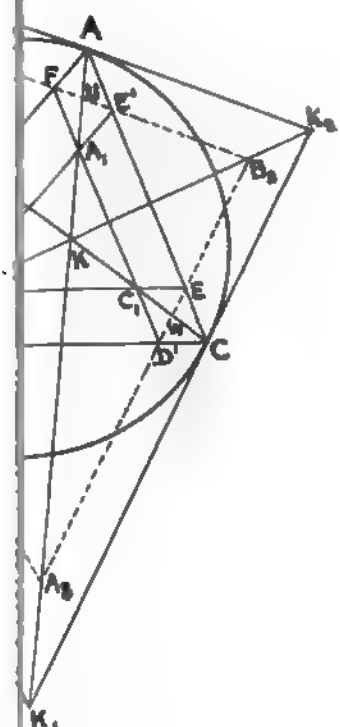
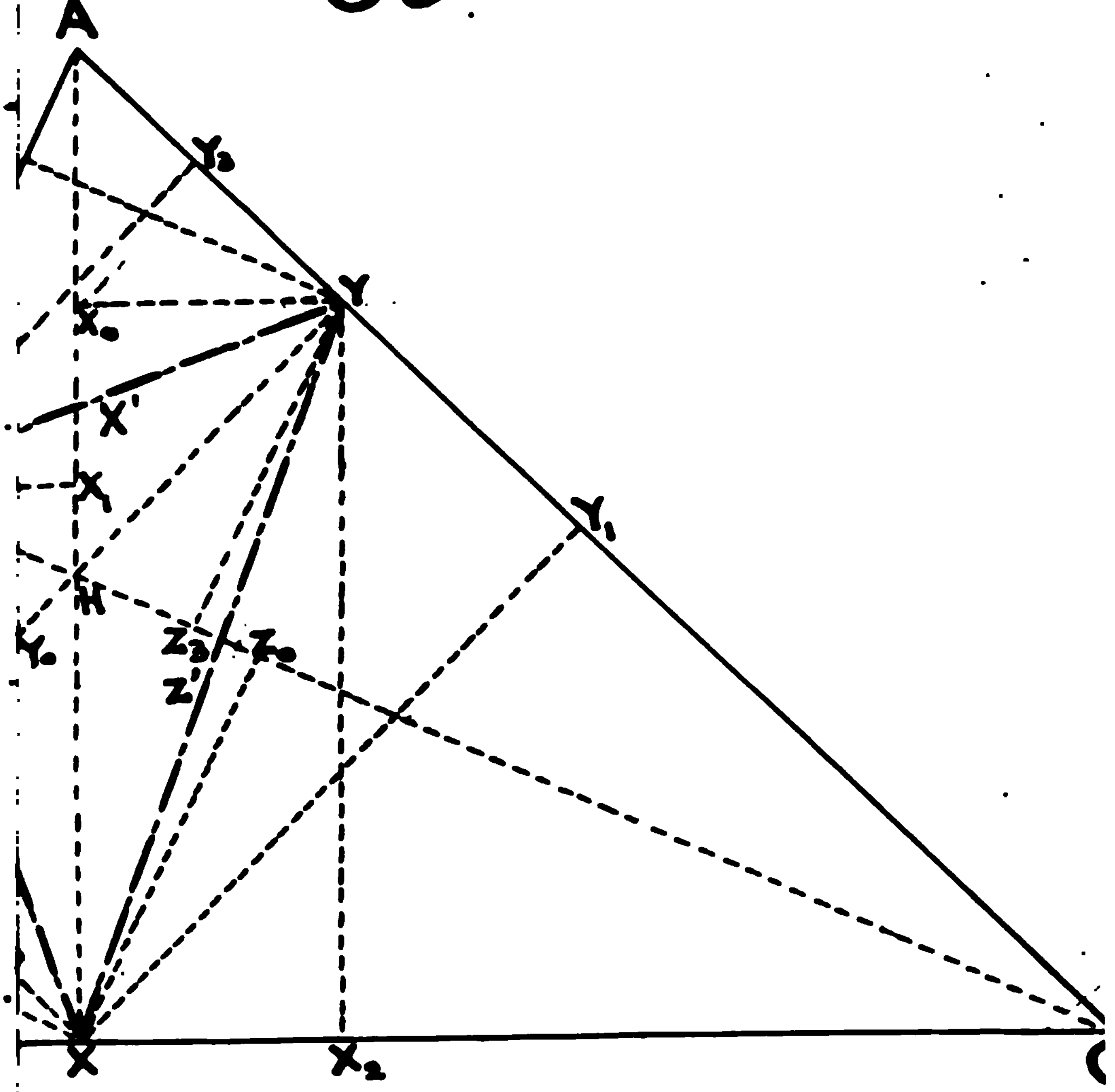

N₂
K₂
C
L₂
M₁


A
F
E'
R
B₃
T
E
P
K
O'
C
D
D'
A₃


A
F
M
E'
K_2
A_1
B_2
K
C_1
E
W
D'
C
A_2
K_1

36



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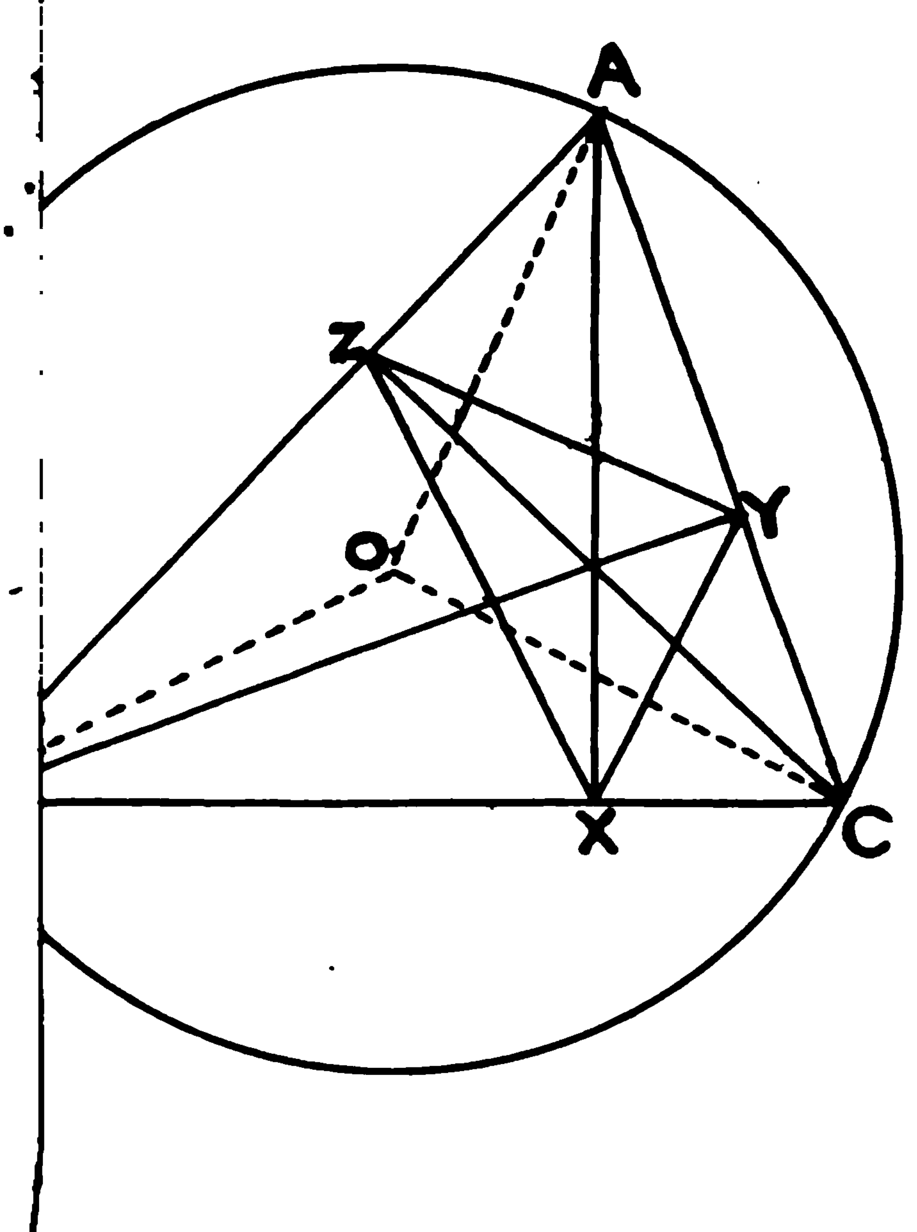

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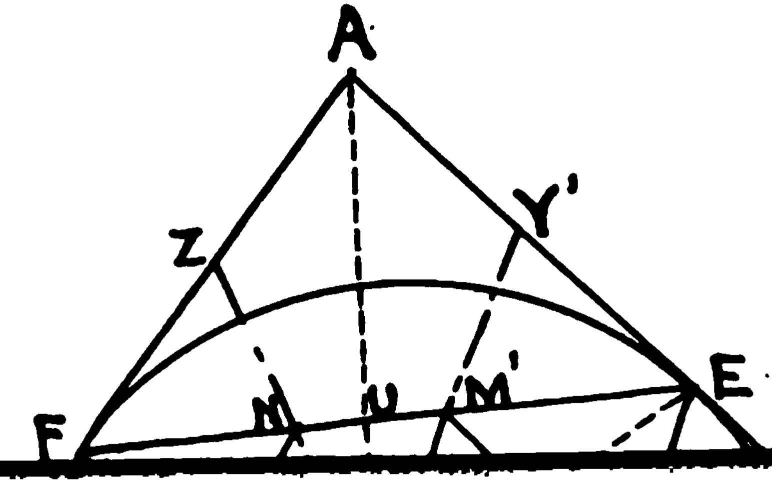

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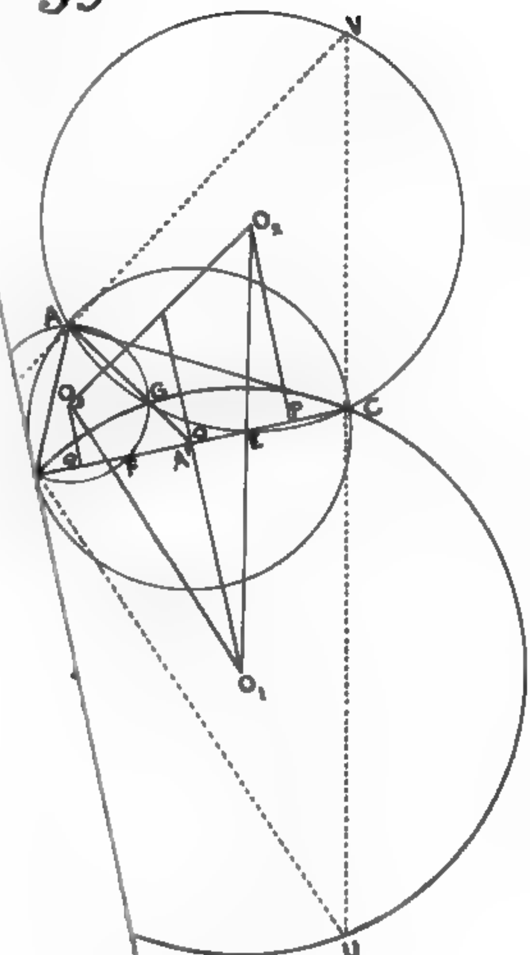

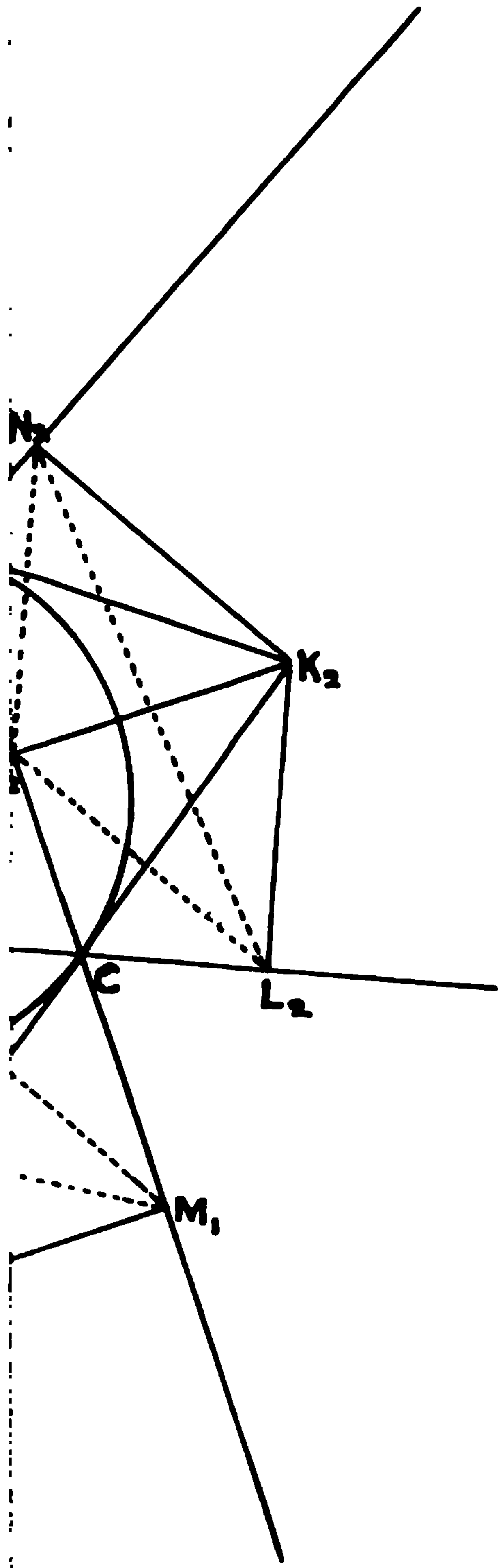

N2
K2
C
L2
M1

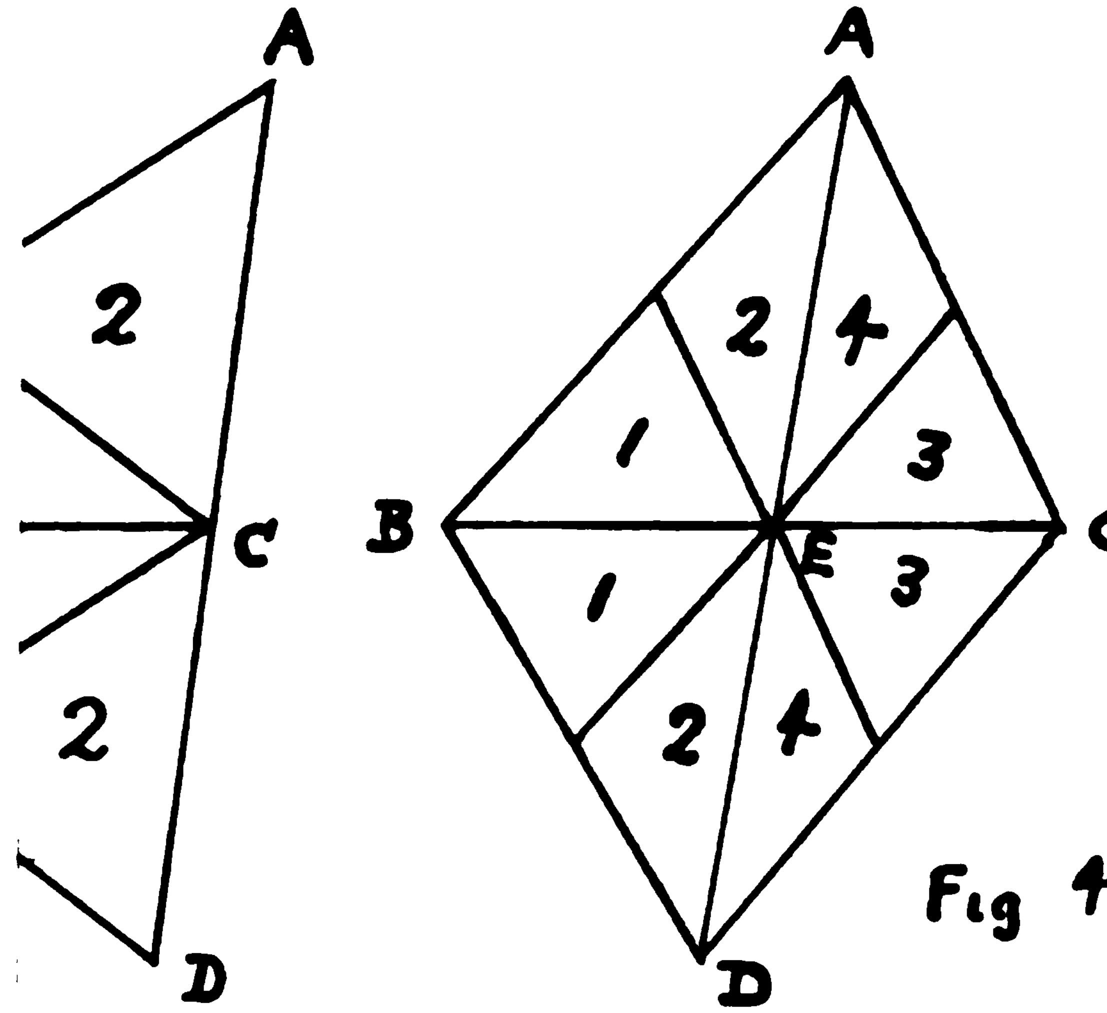

Fig 41

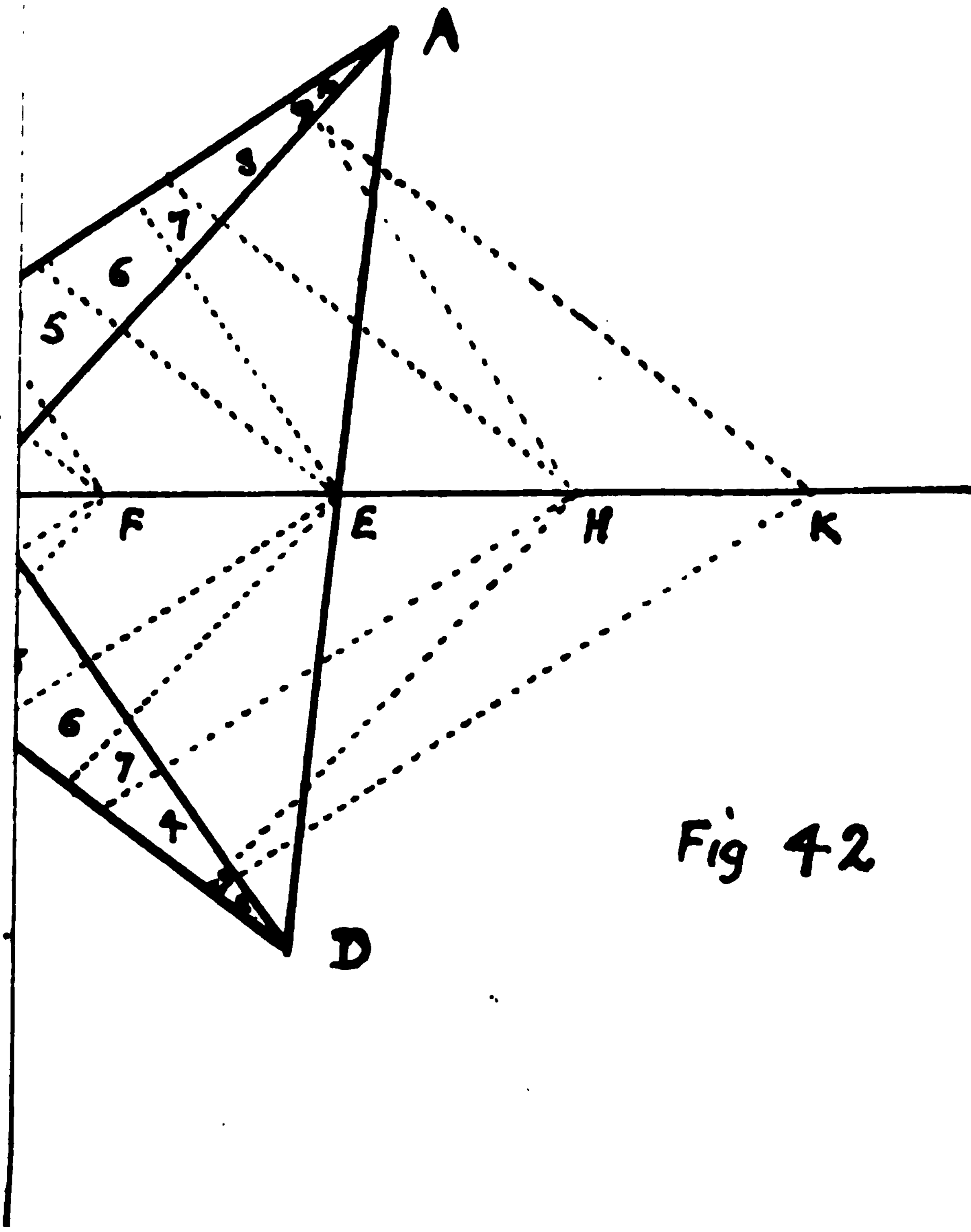

Fig 42

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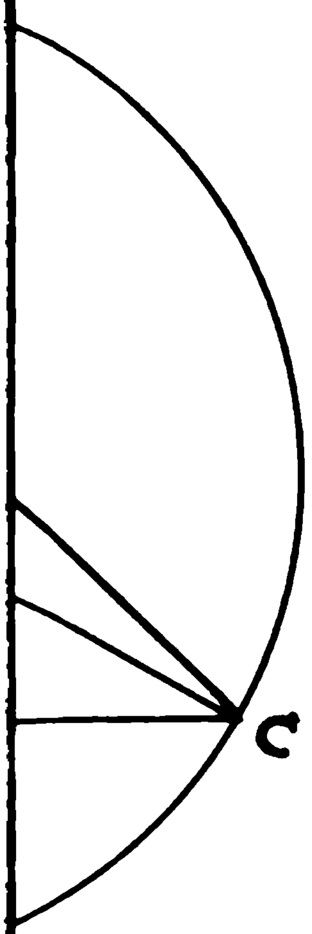

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Fig 44

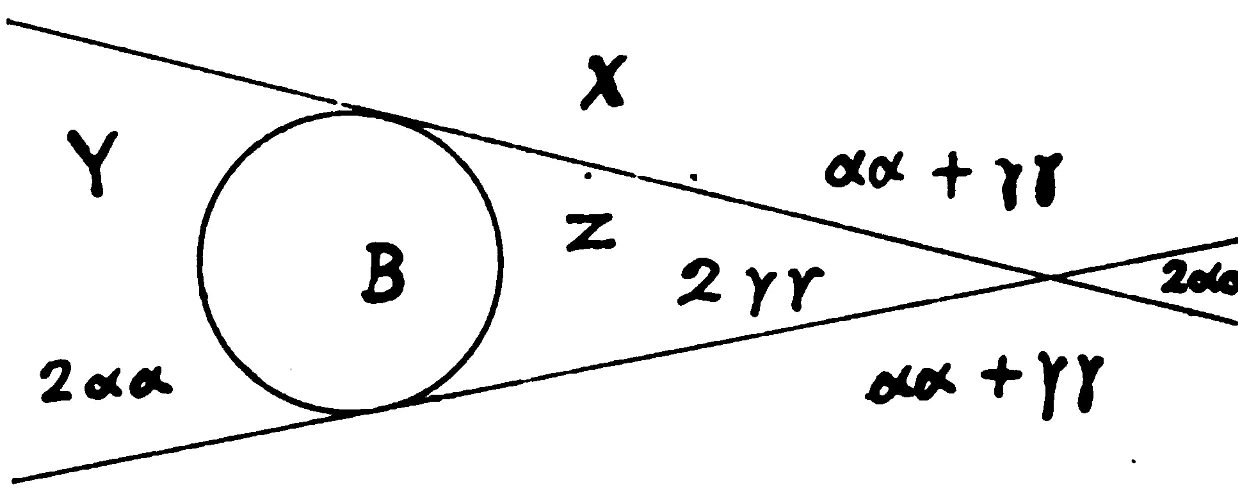


Fig 45

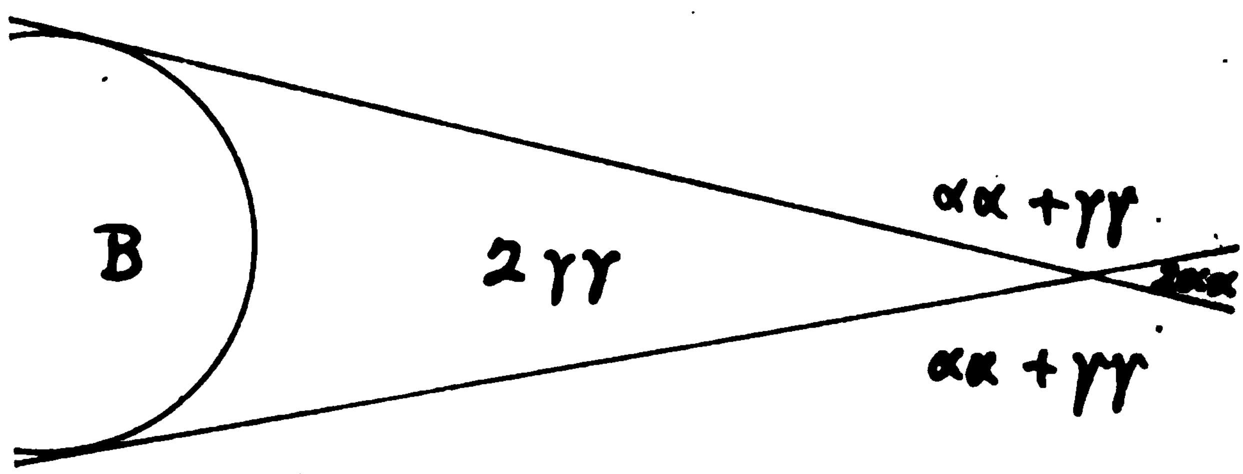
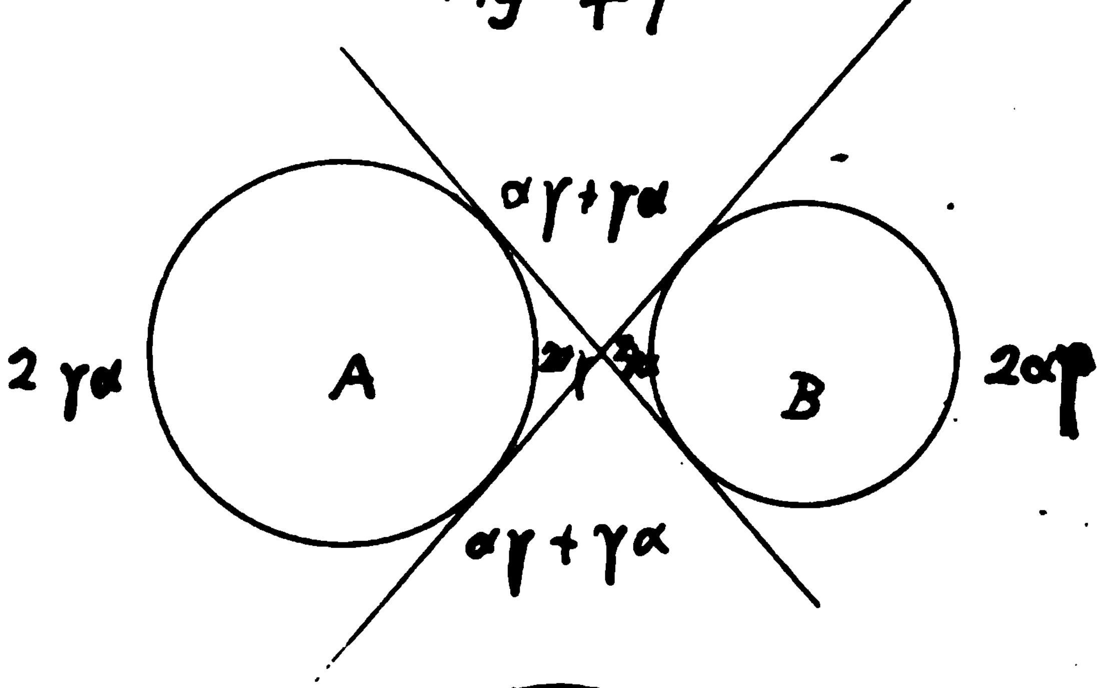


Fig 47



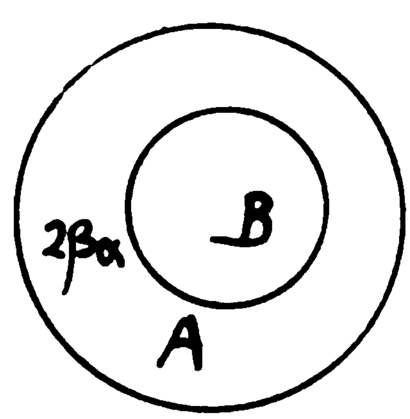
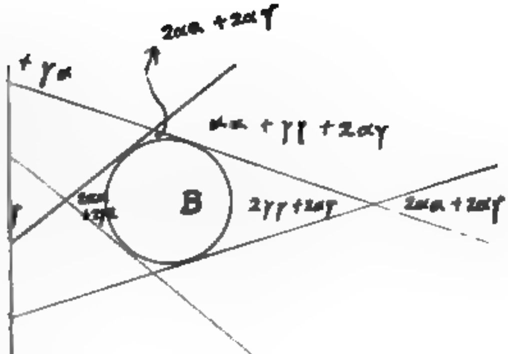


Fig 49

1 50



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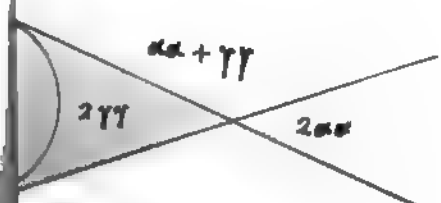

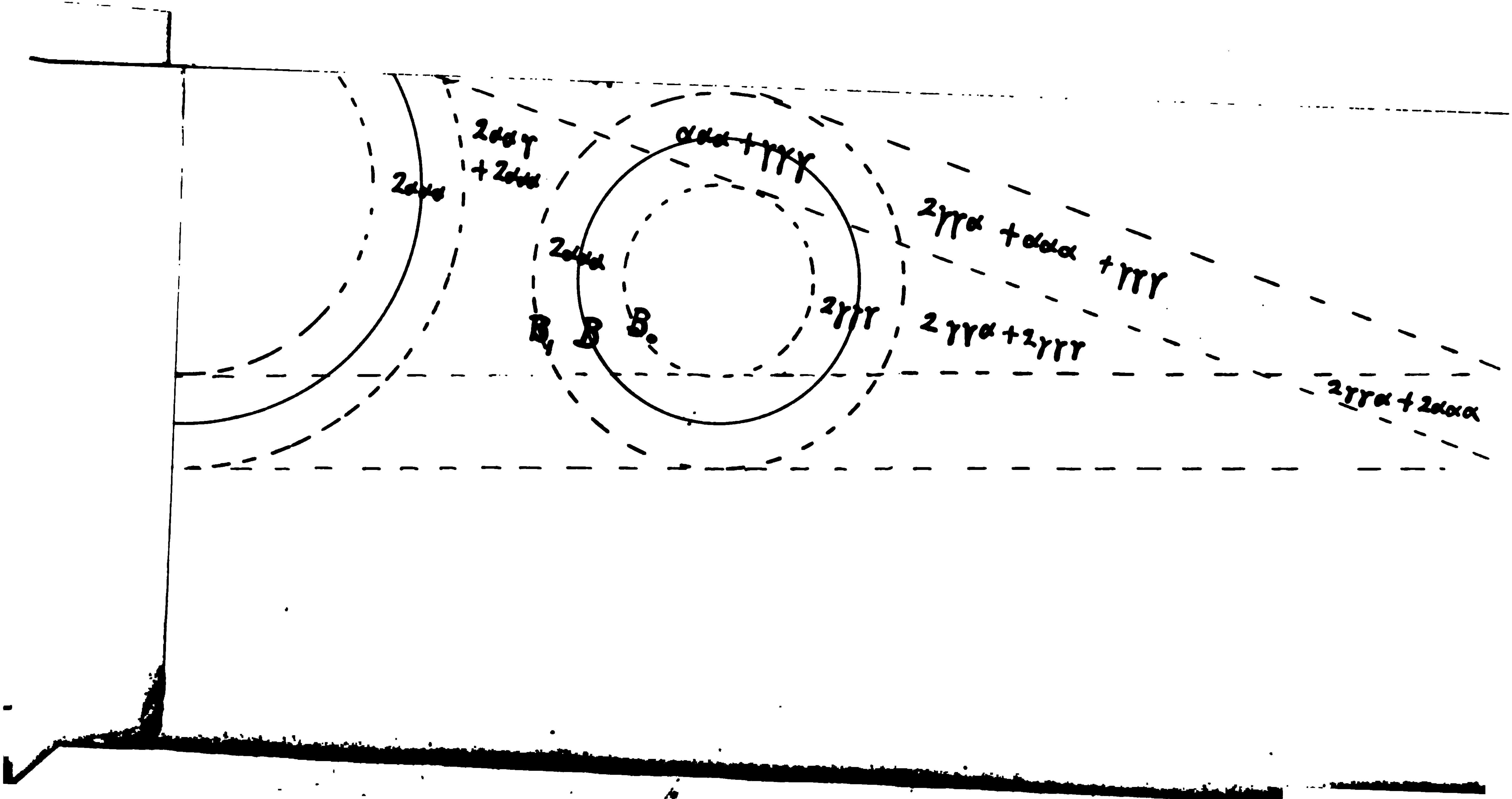
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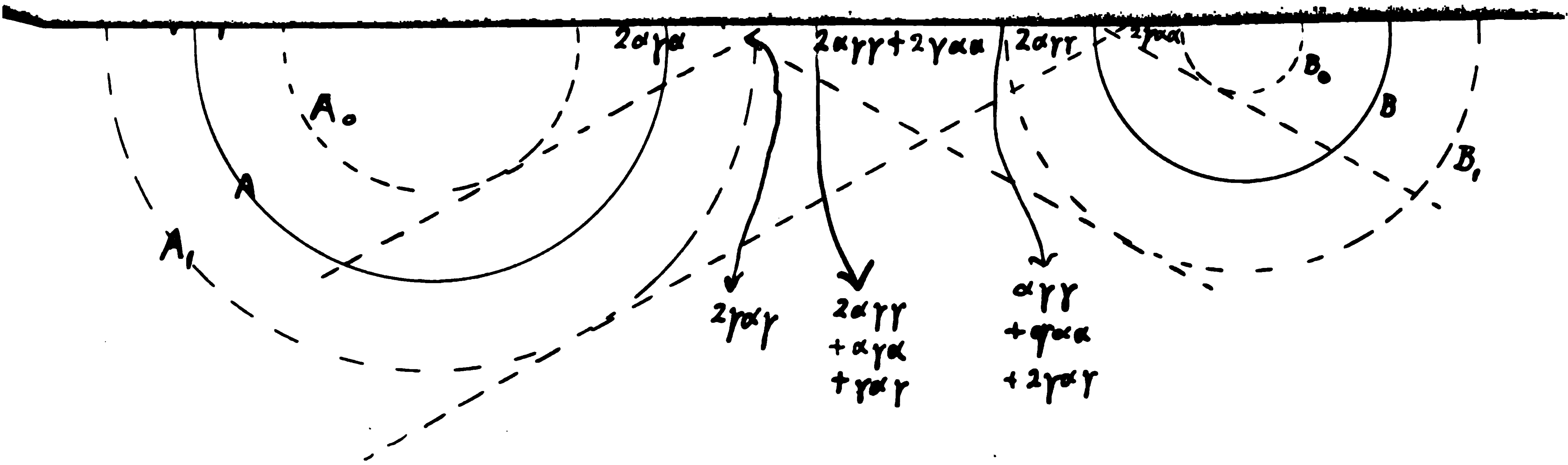
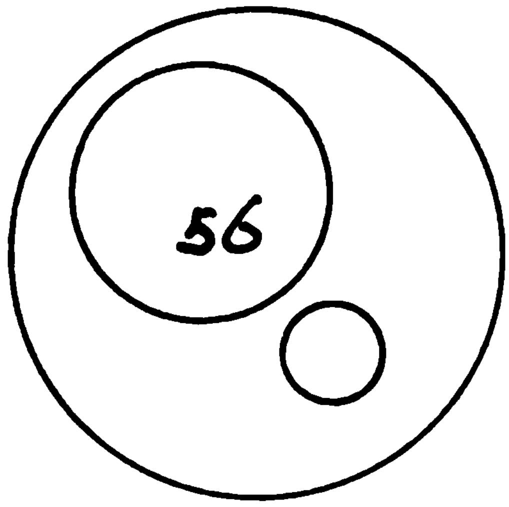
+ 2ααα
2ααα
αγα + γγγ
2ααα
B_1
B
B_e
2γγγ
2γγα + ααα + γγγ
2γγα + 2γγγ
2γγα + 2ααα



A₀
A
A₁
B₀
B
B₁
2αγα
2αγγ+2γαα
2αγγ
2γαα
2γαγ
2αγγ
+αγα
+γαγ
αγγ
+2γαγ

Fig 54.



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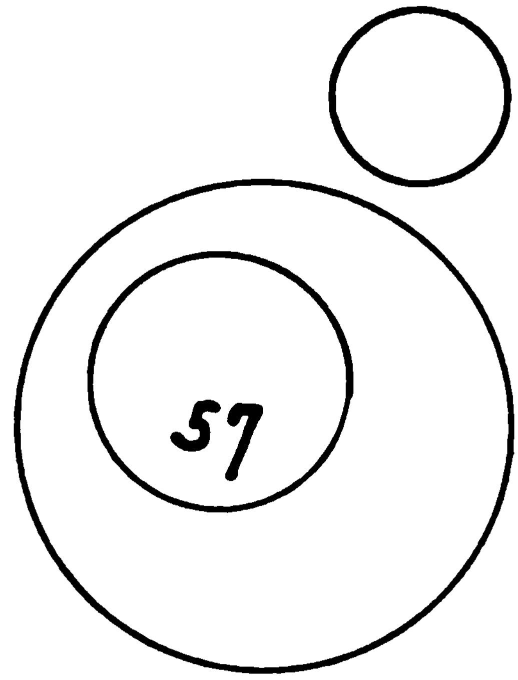

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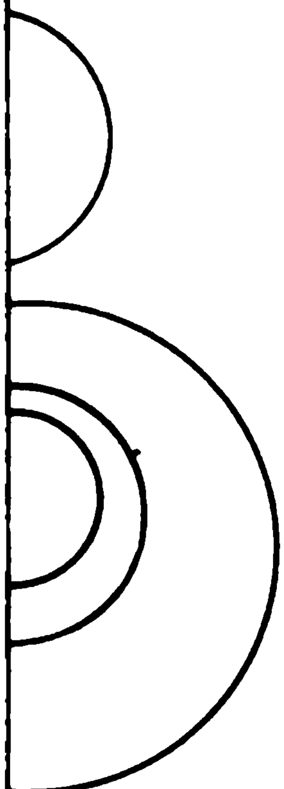

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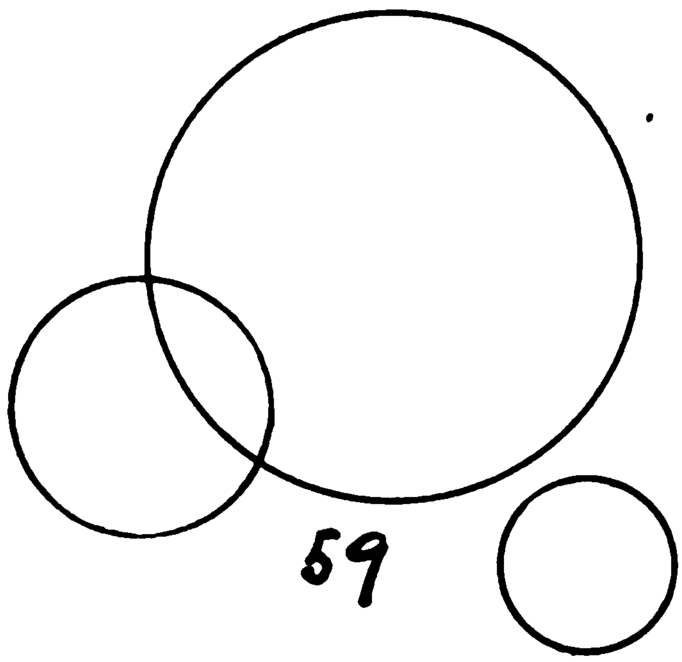

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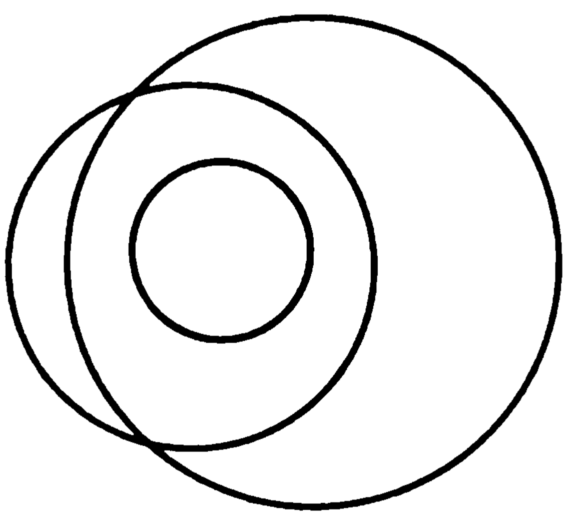

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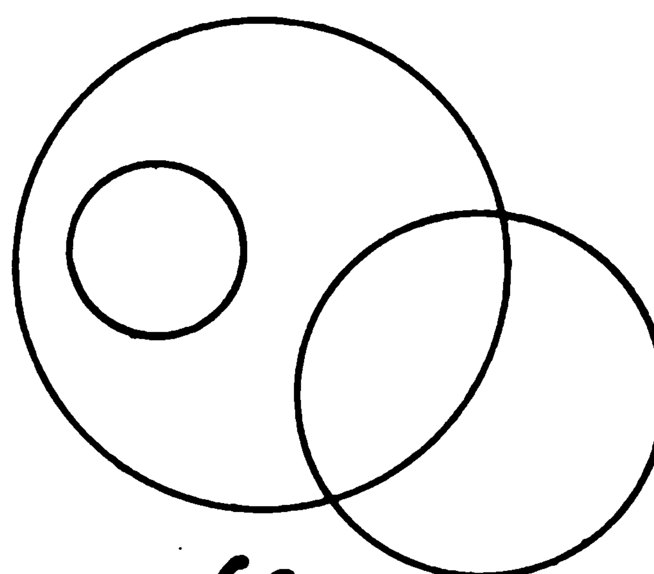

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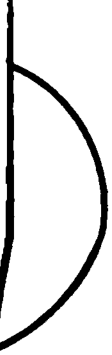

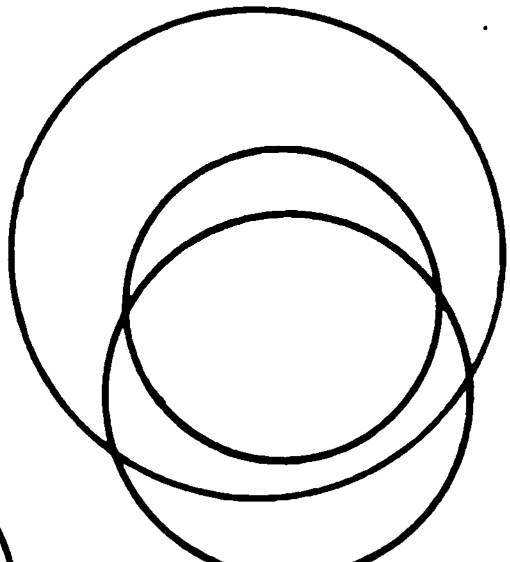

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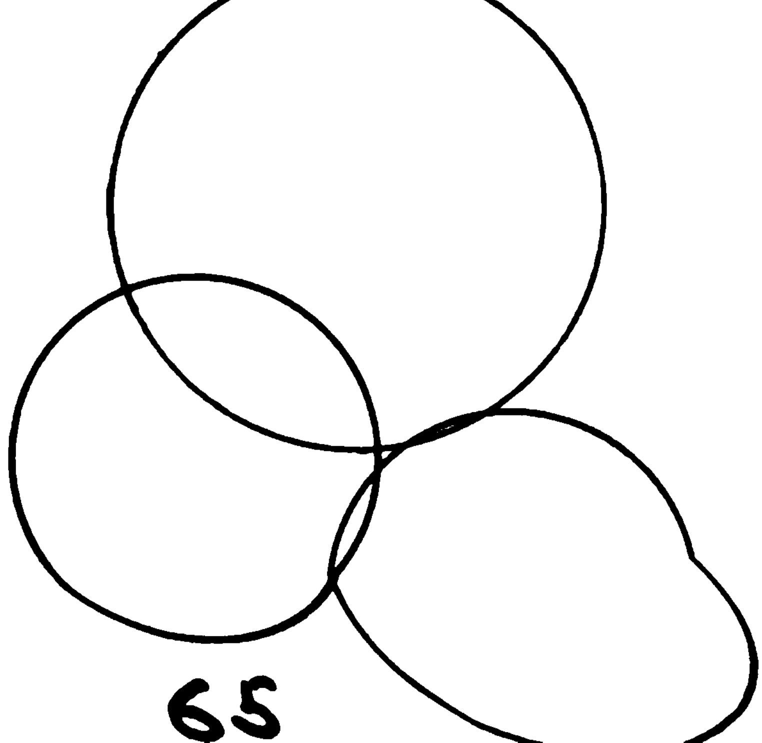

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Fig 44

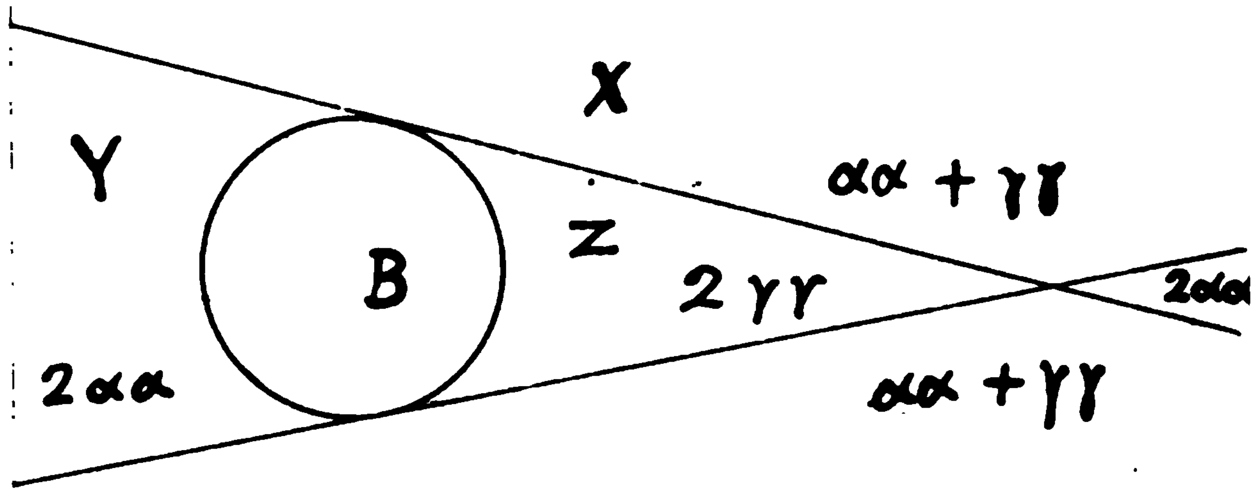


Fig 45

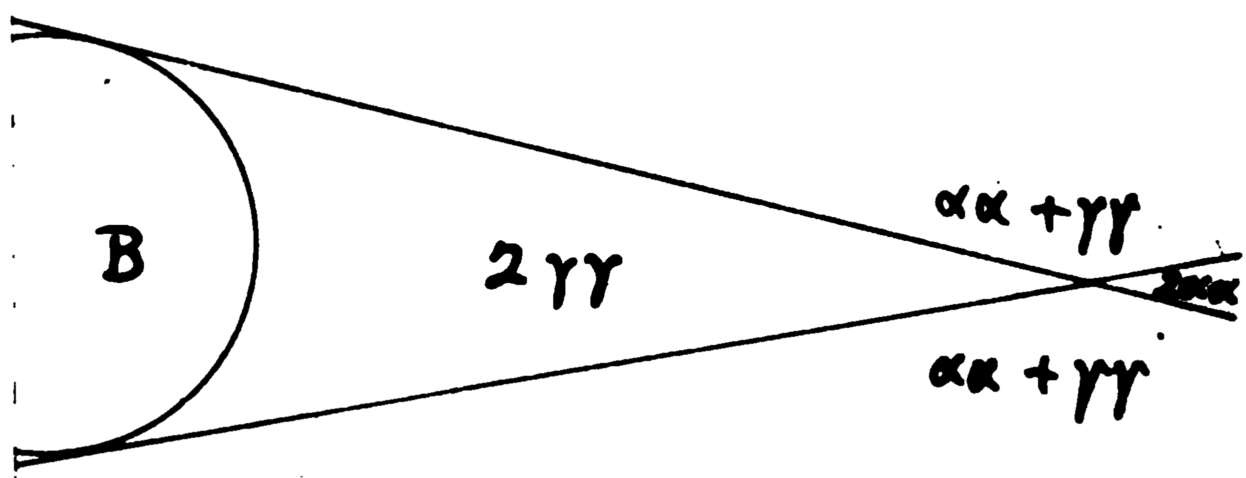


Fig 47

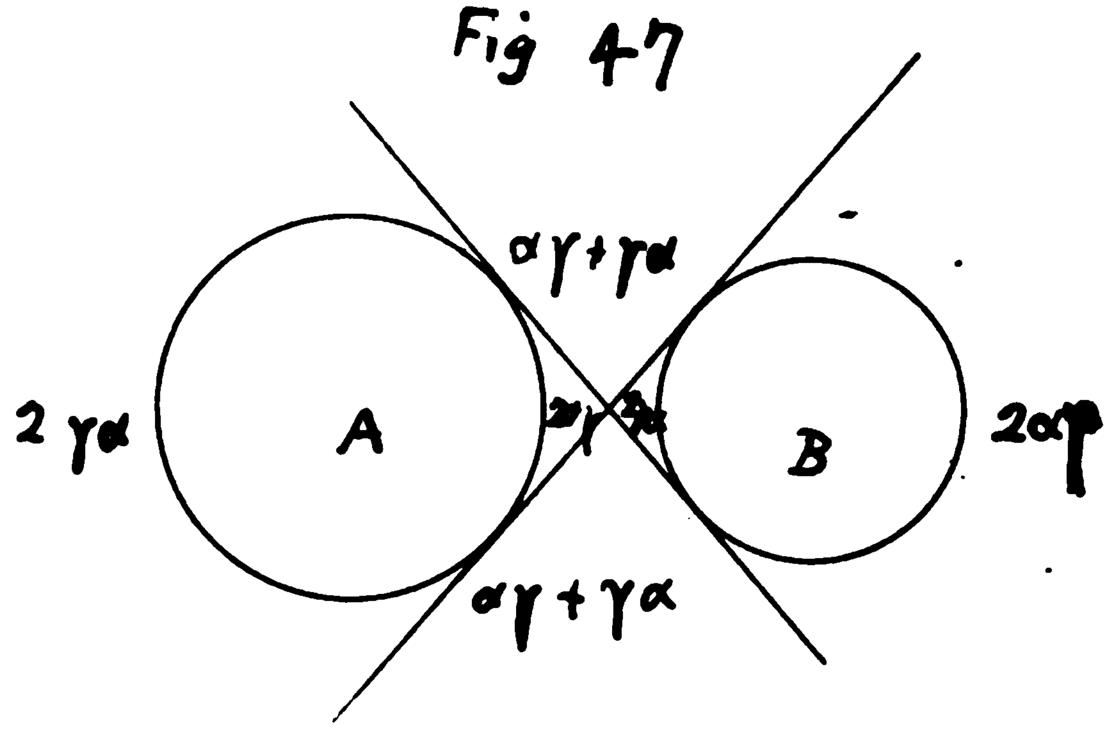
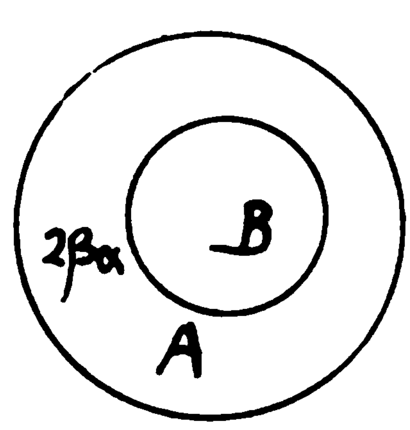
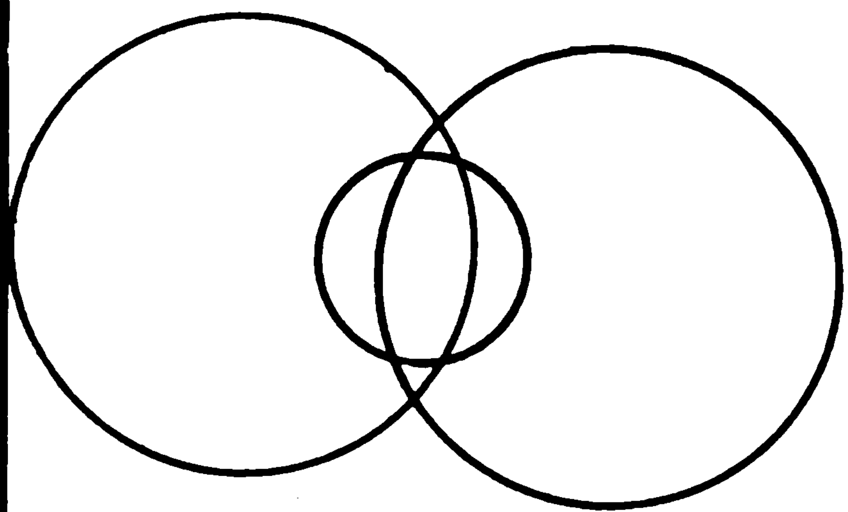


Fig 49





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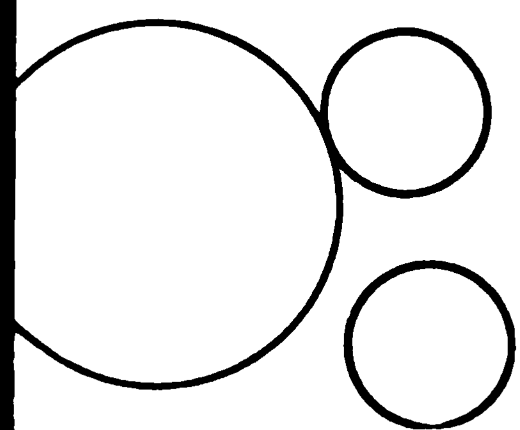

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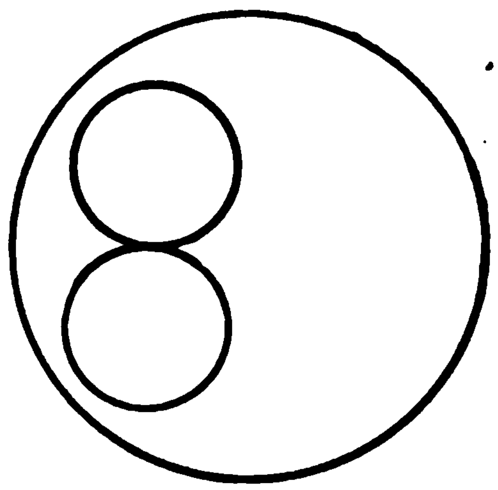

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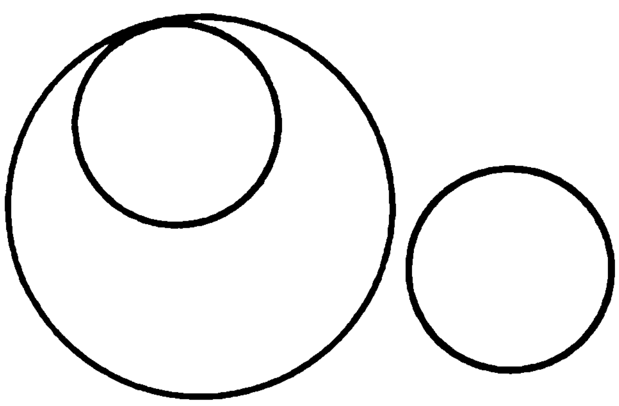

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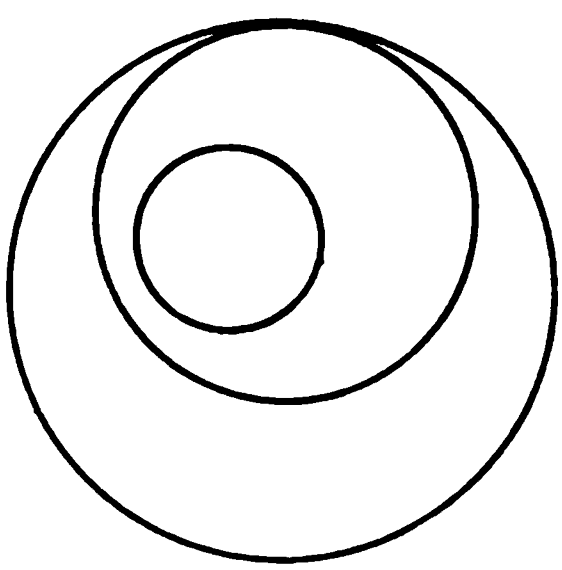

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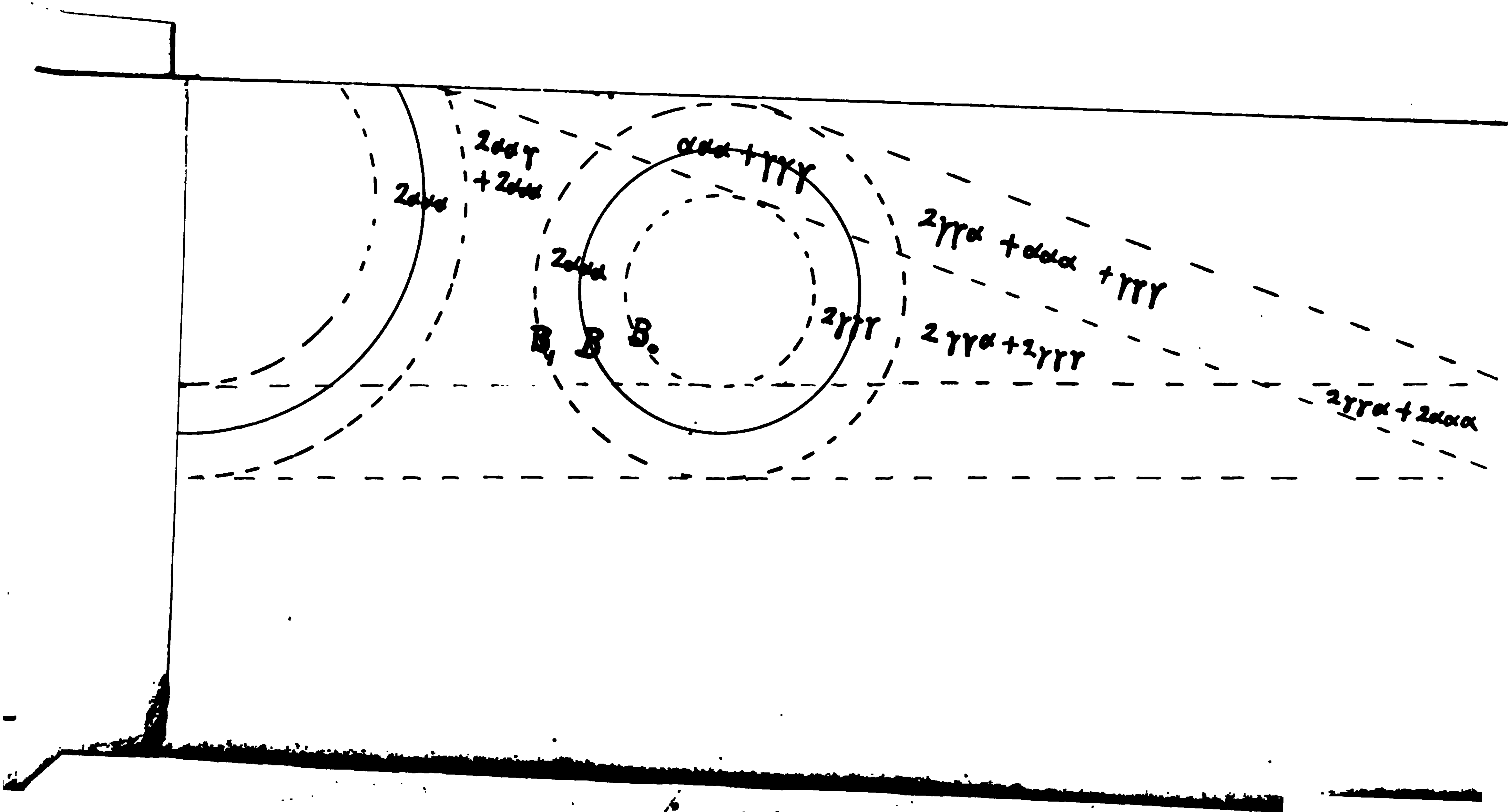
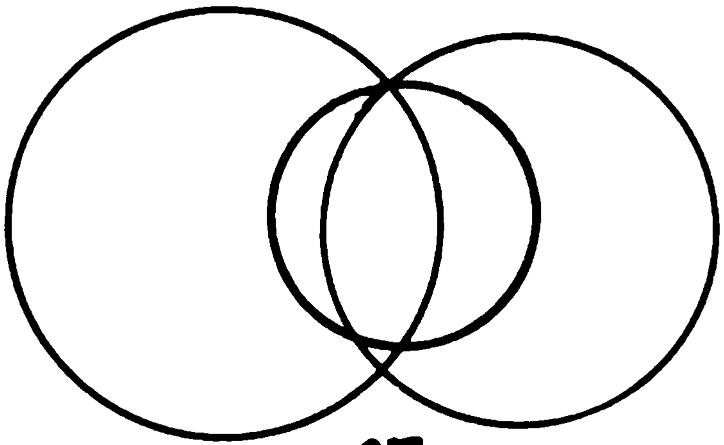


2ααγ + 2ααα
2ααα
ααα + γγγ
2ααα
B_1
B
B_0
2γγγ
2γγα + ααα + γγγ
2γγα + 2γγγ
2γγα + 2ααα



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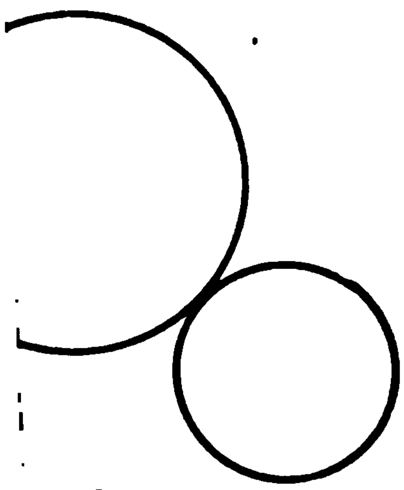

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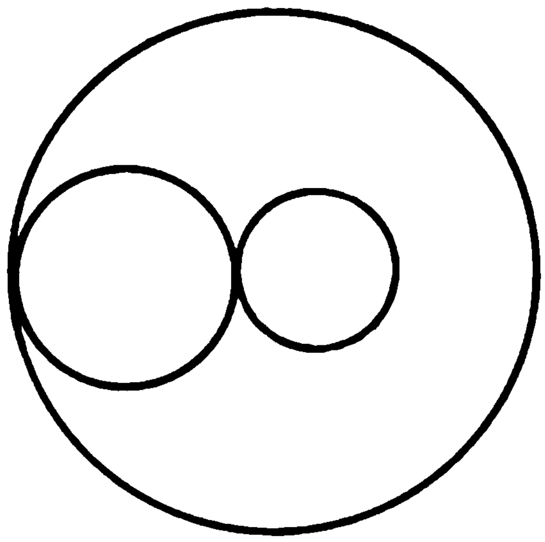

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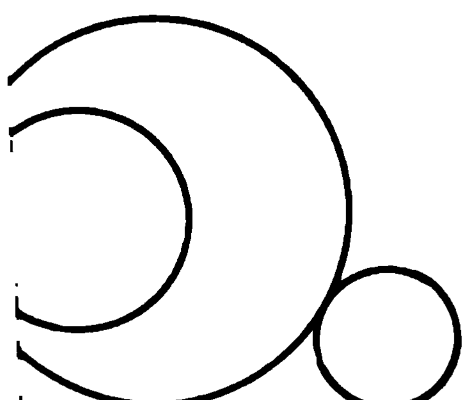

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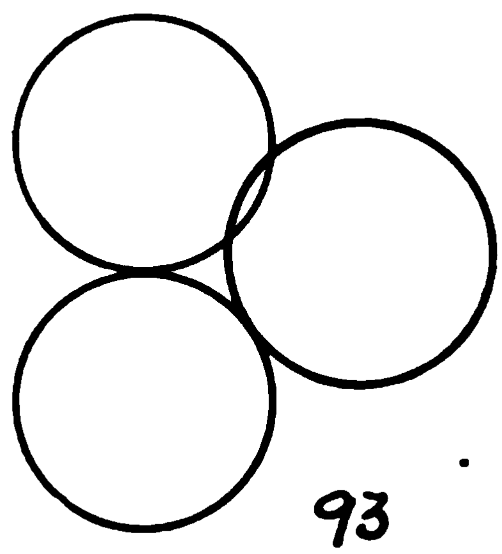

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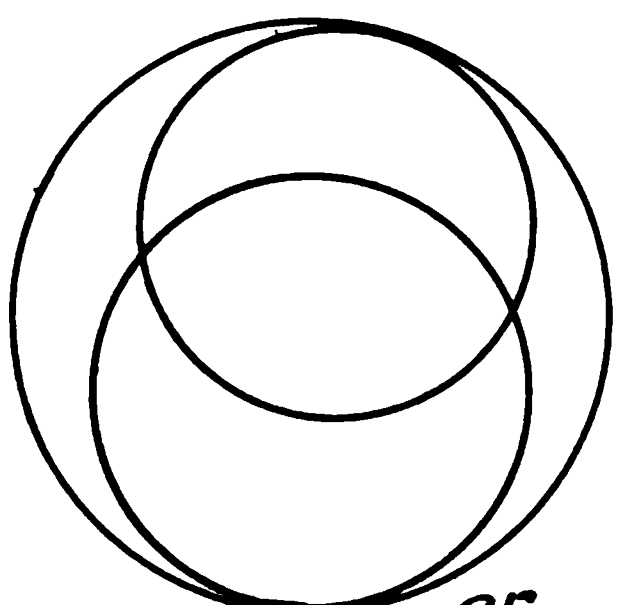

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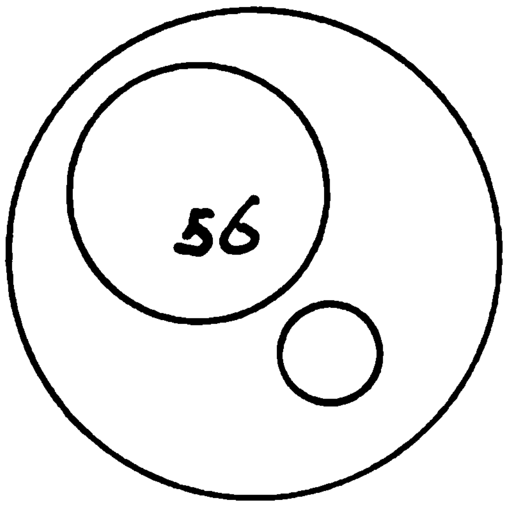

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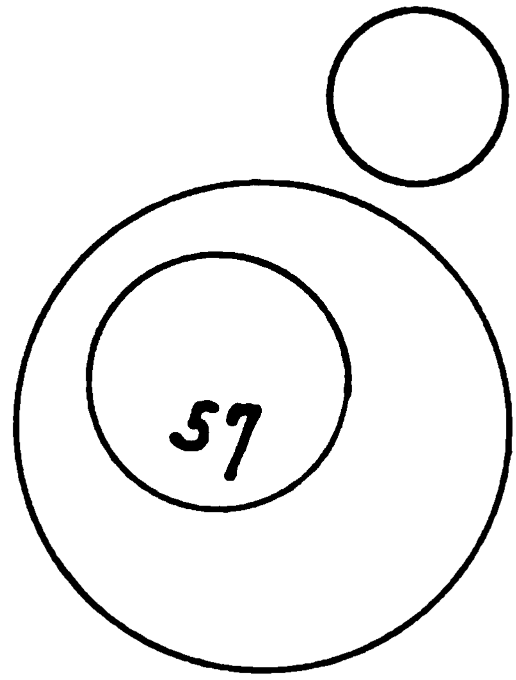

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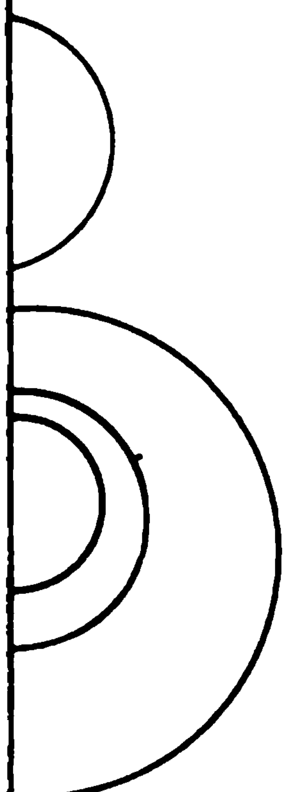

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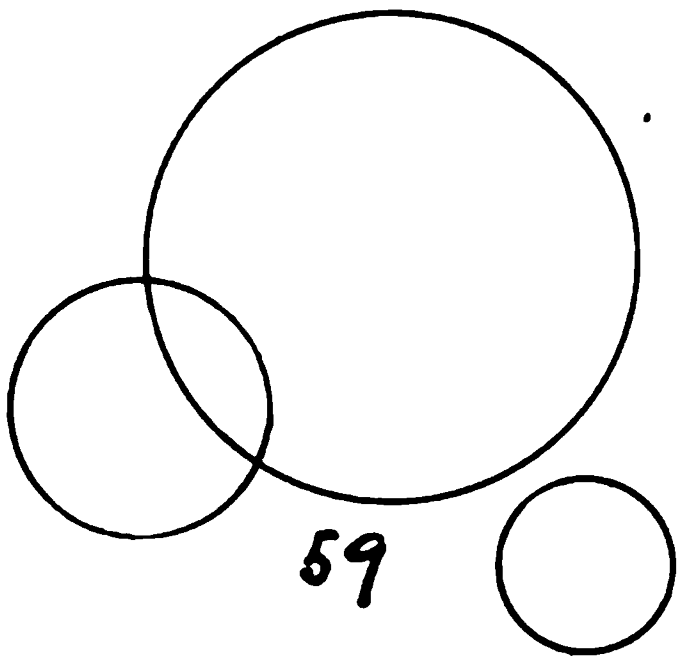

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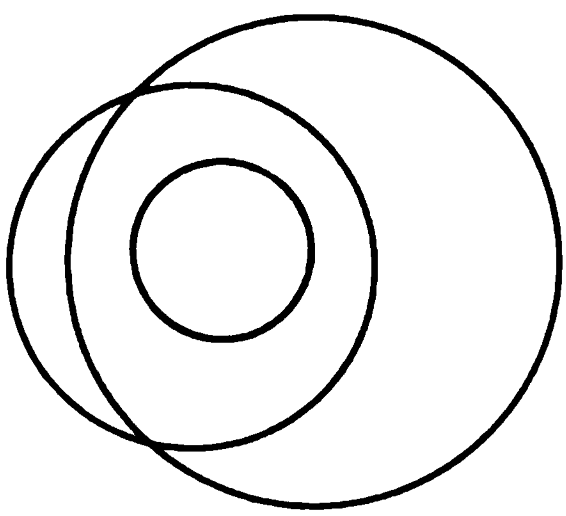

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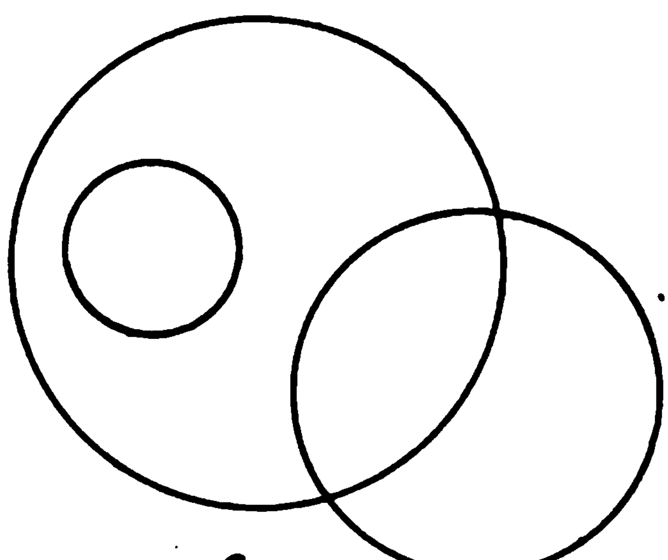

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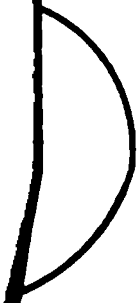

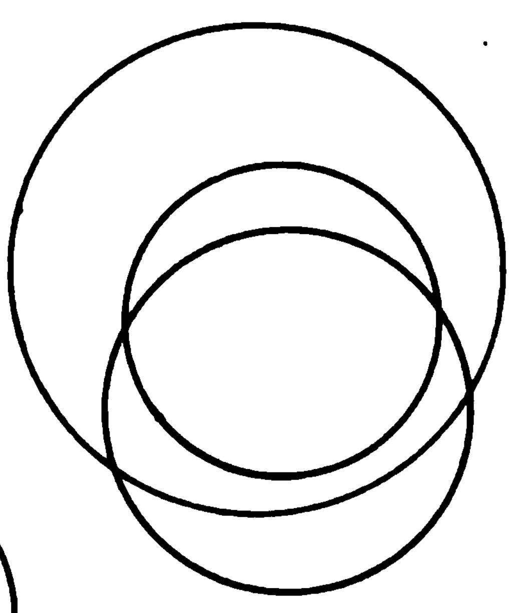

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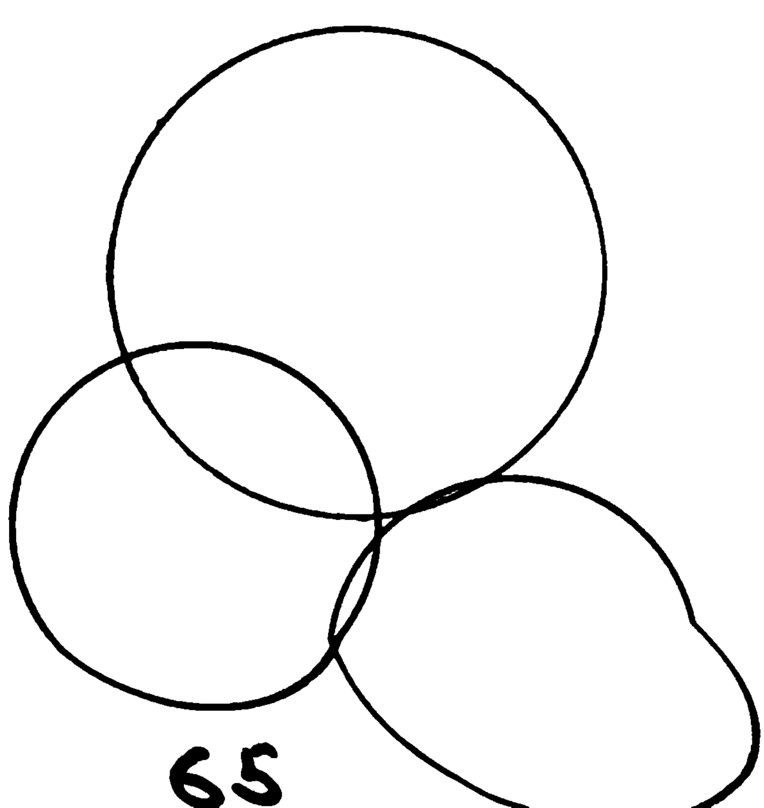

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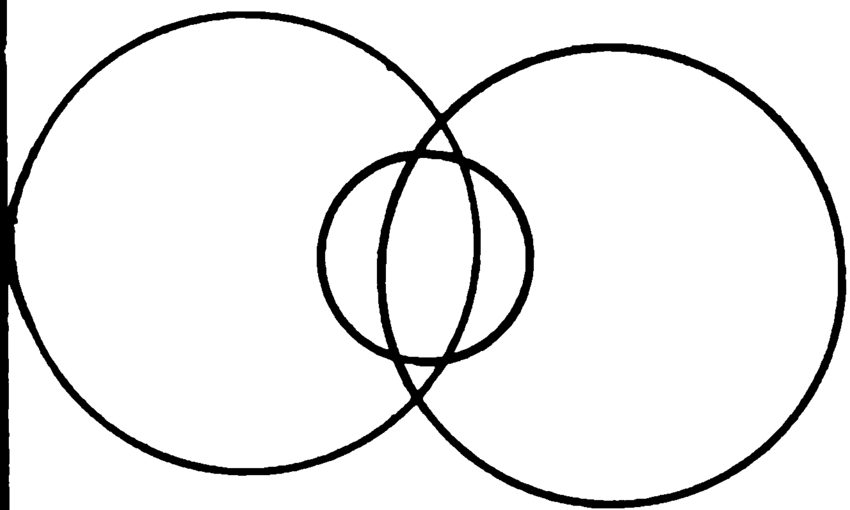

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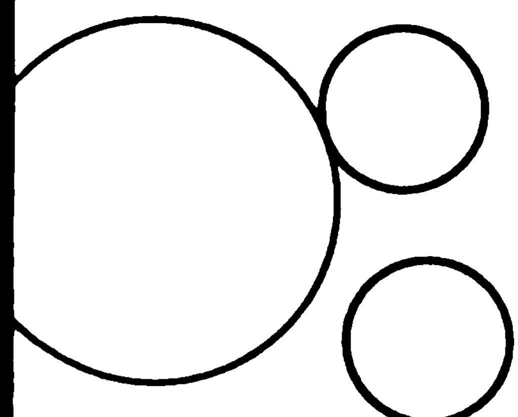

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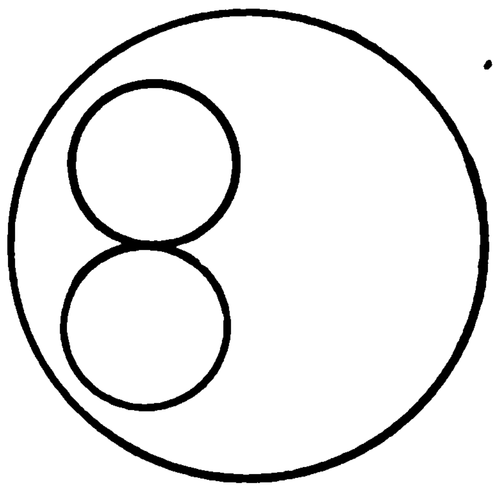

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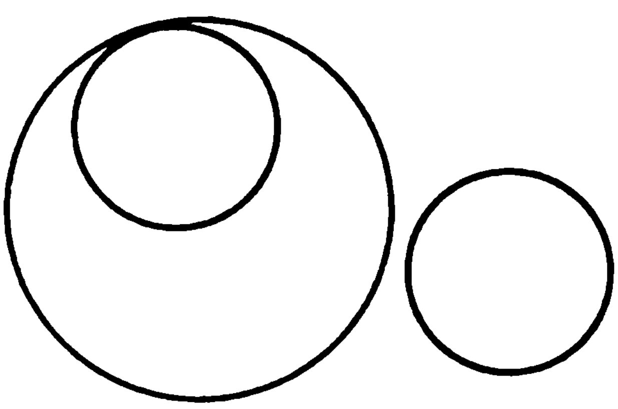

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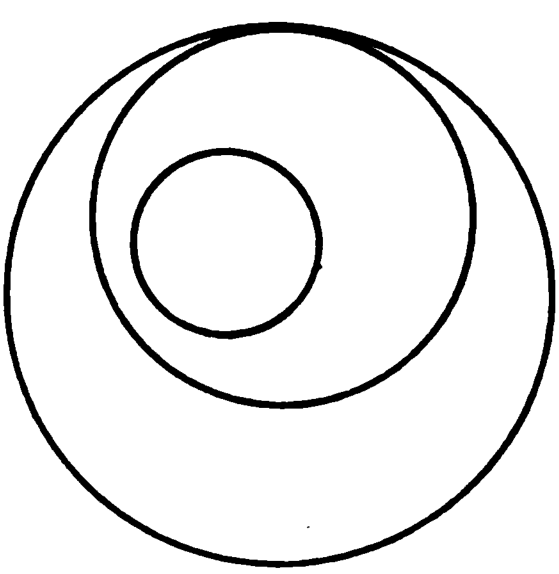

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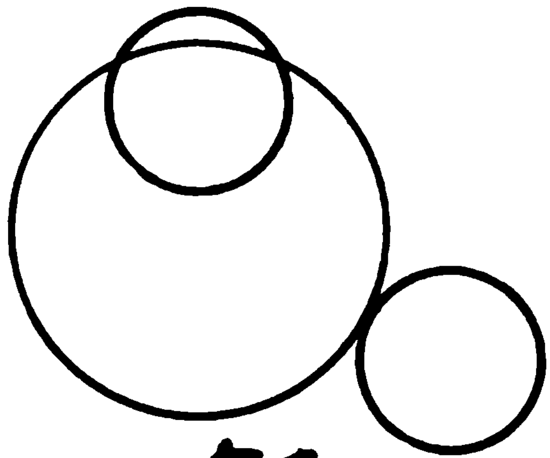

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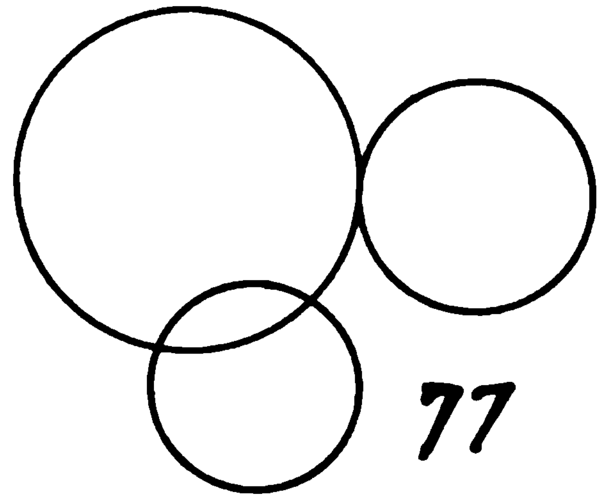

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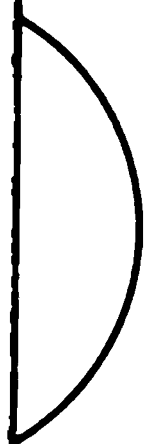

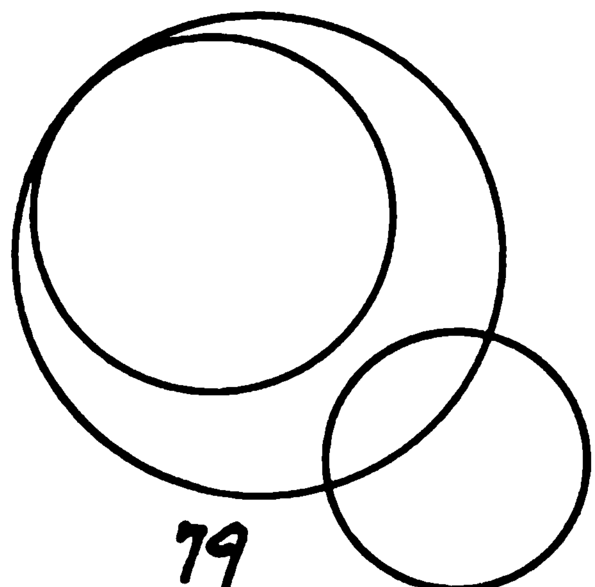

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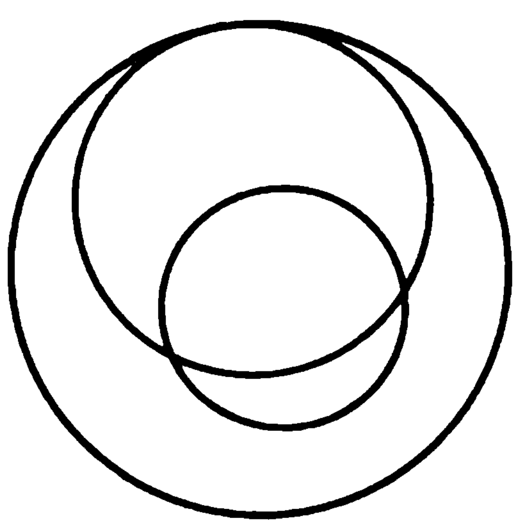

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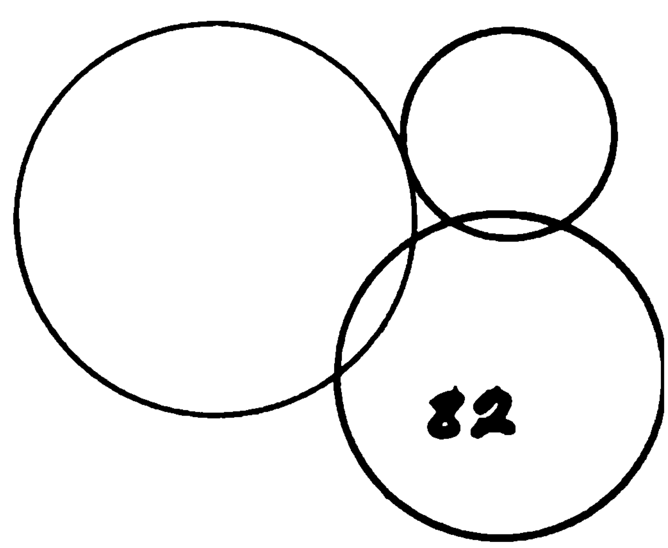

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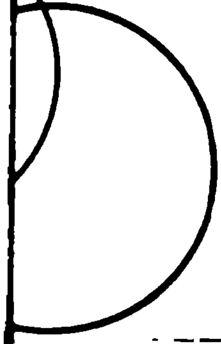

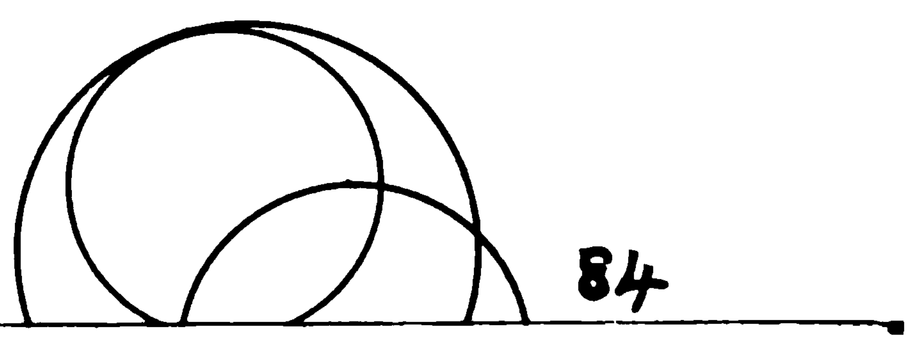

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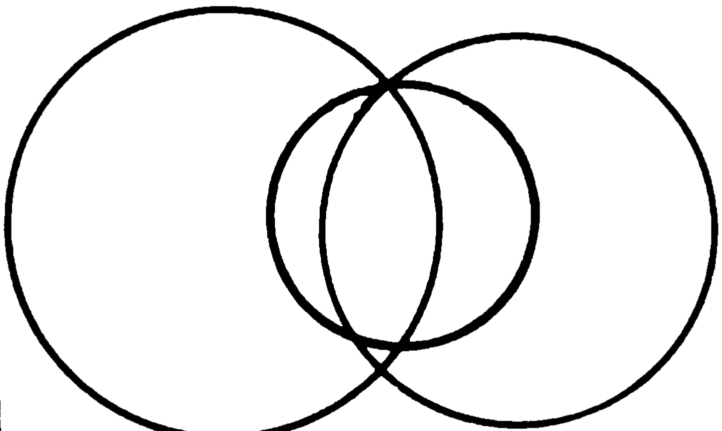

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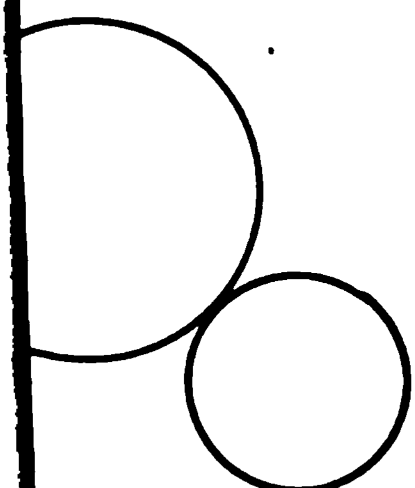

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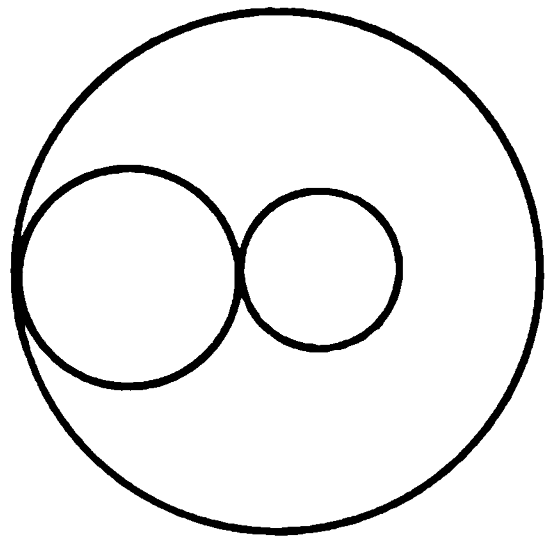

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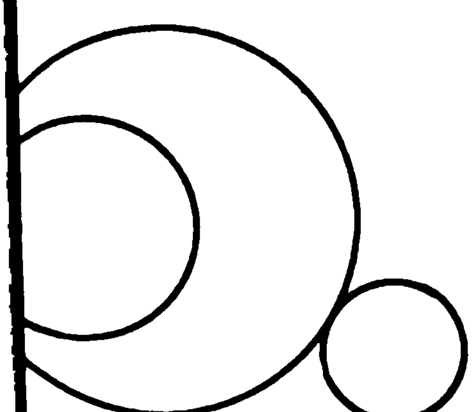

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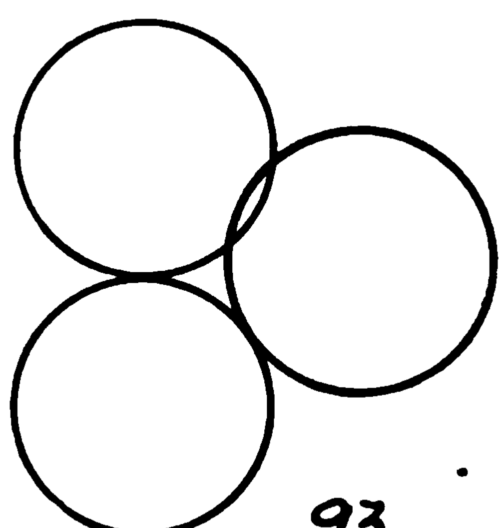

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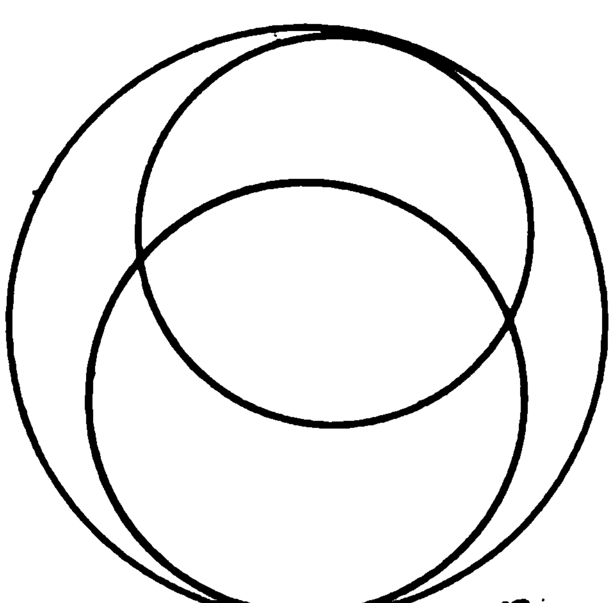

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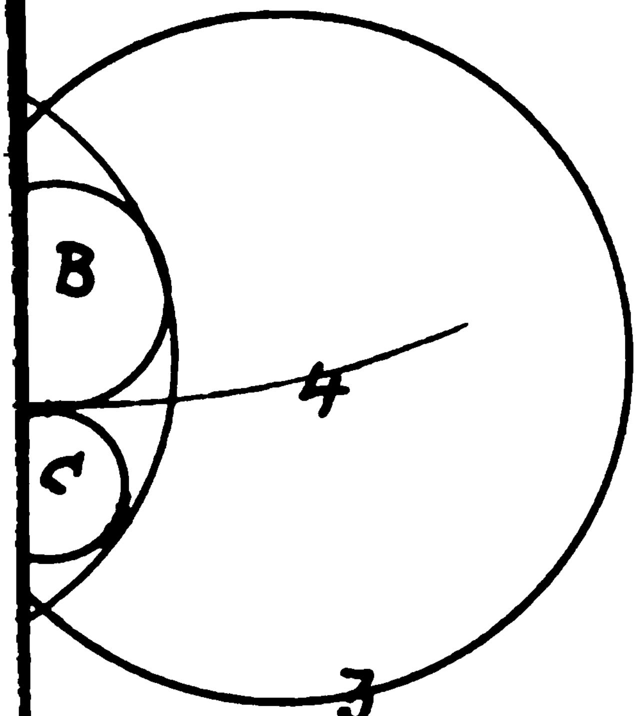

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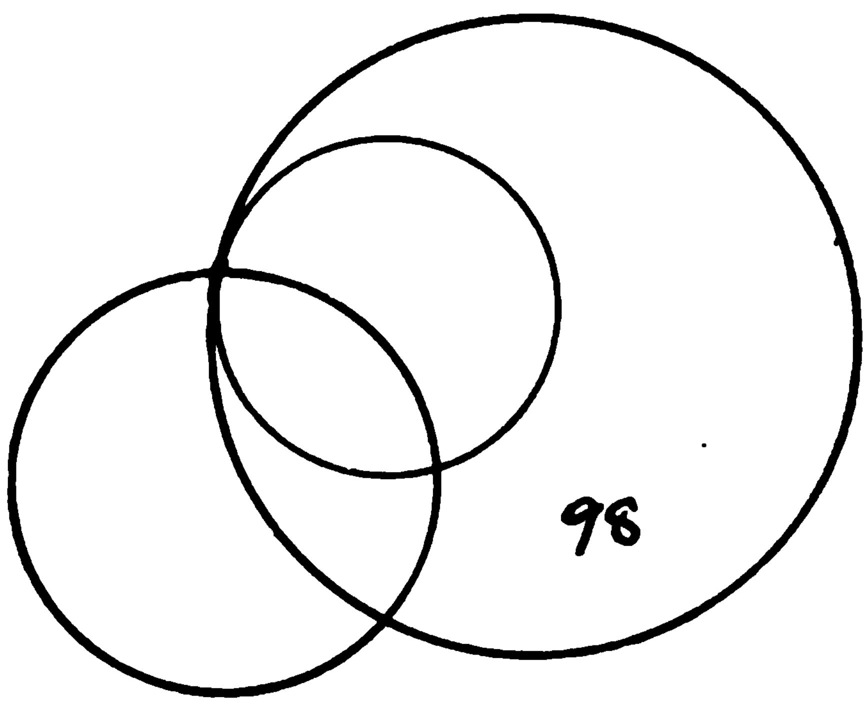
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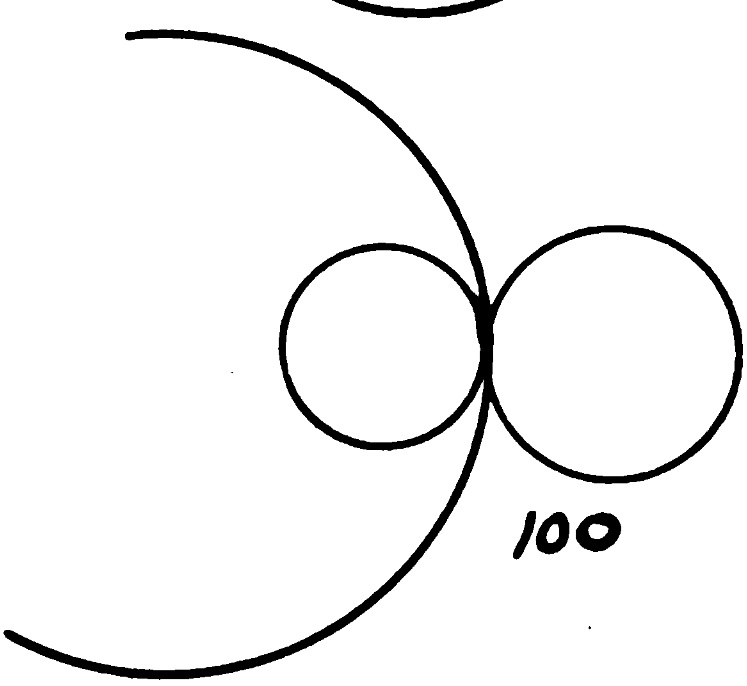
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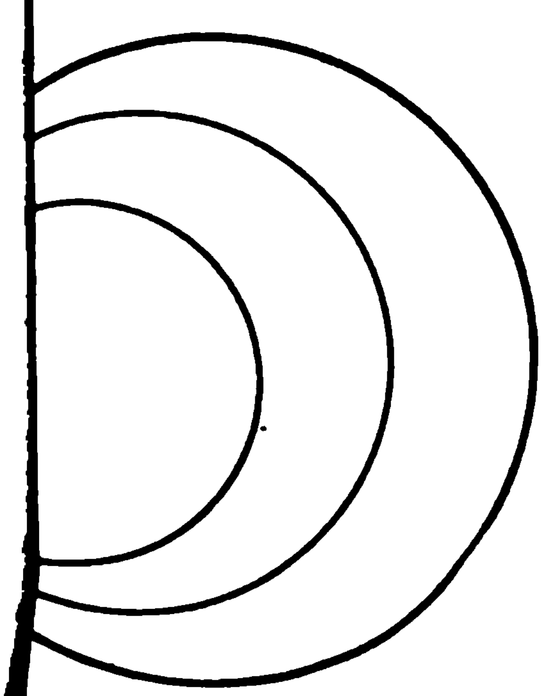
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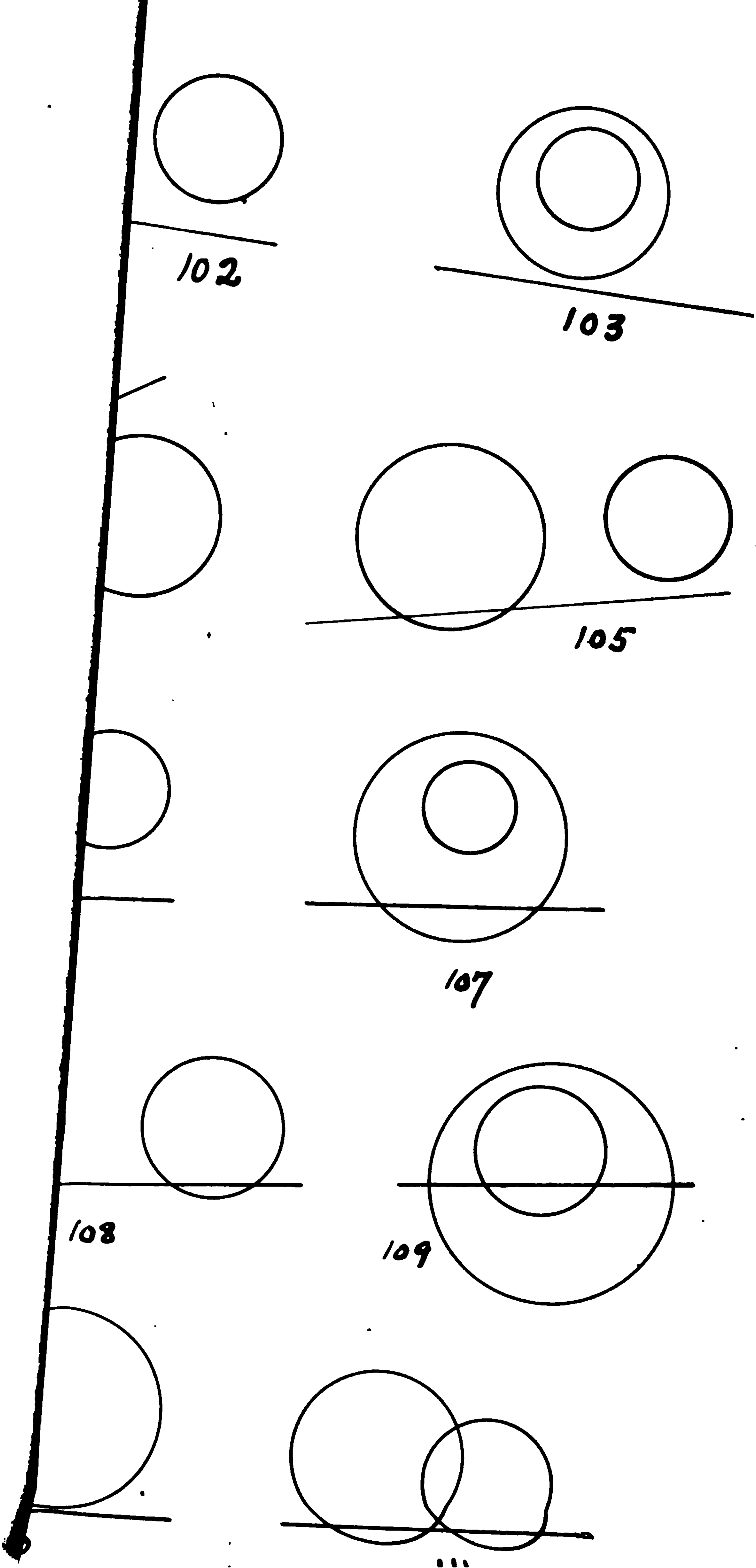

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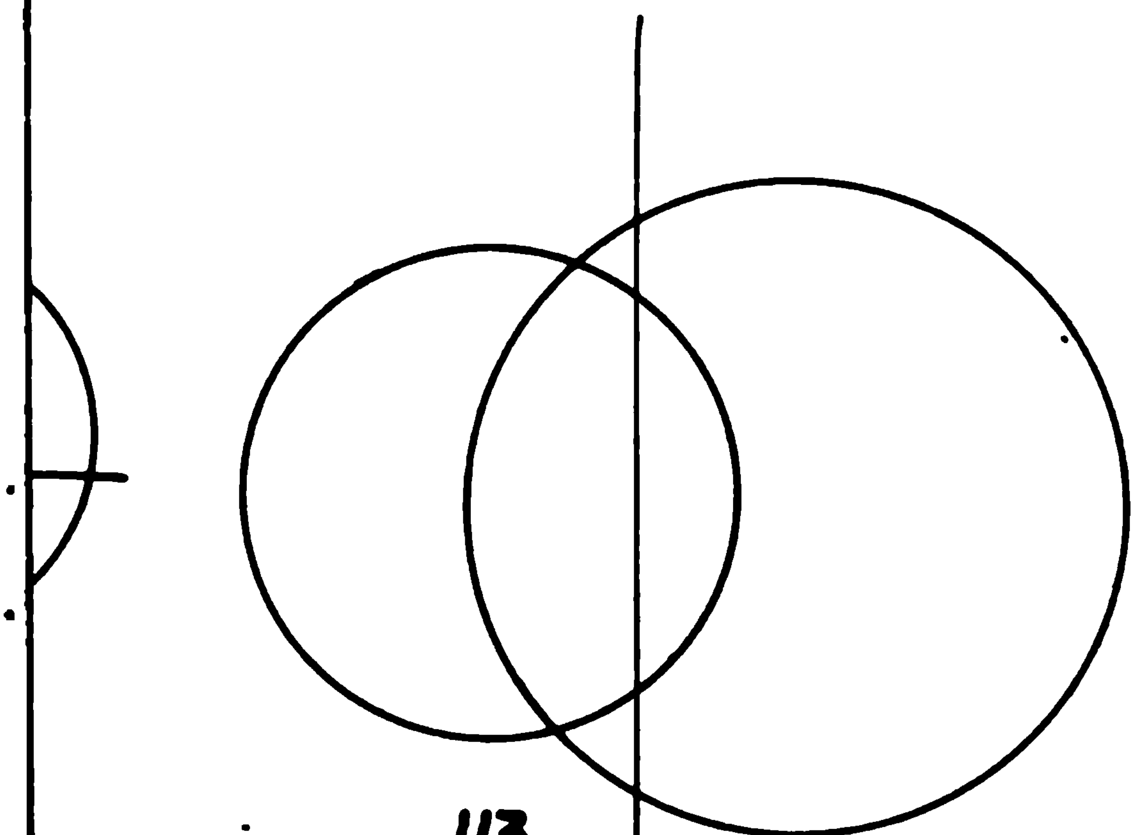

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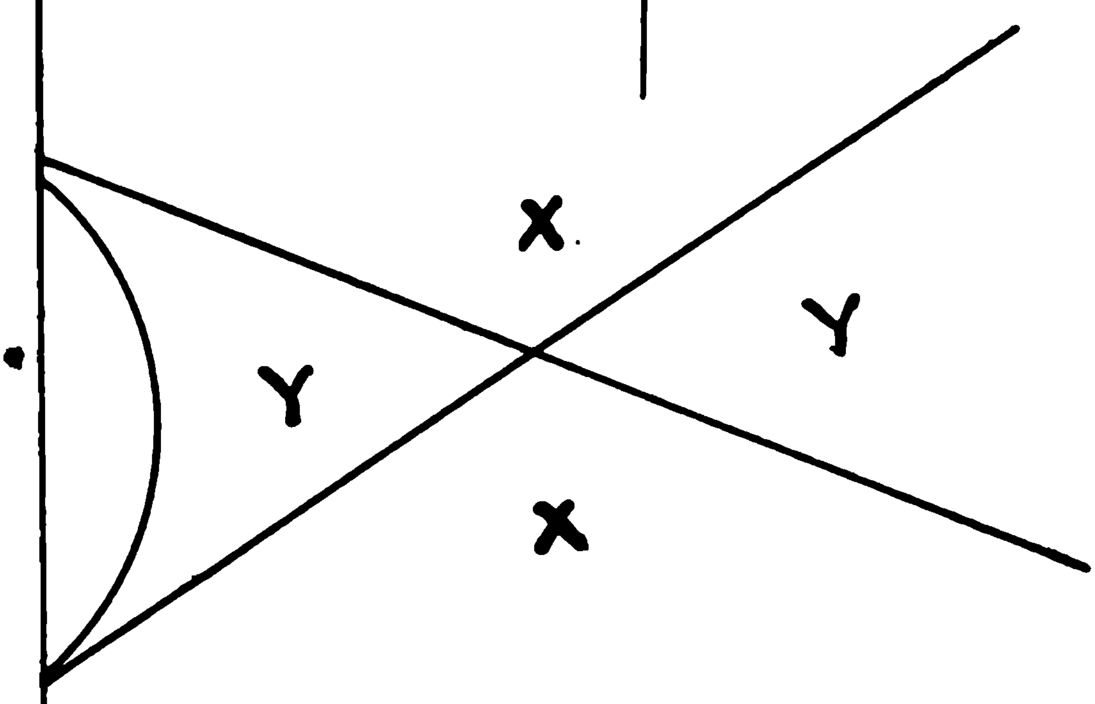

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